Proof Theory of Modal Logic

Lecture 1
Sequent Calculus and Modal Logic

Marianna Girlando

ILLC, Universtiy of Amsterdam

5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Practical information

Practical information

- Who?
Marianna Girlando (m.girlando@uva.nl)
Sisi Yang (yangss23@mails.tsinghua.edu.cn)
Xin Li (lixin24@mails.tsinghua.edu.cn)

Practical information

- Who?
  Marianna Girlando (`m.girlando@uva.nl`)
  Sisi Yang (`yangss23@mails.tsinghua.edu.cn`)
  Xin Li (`lixin24@mails.tsinghua.edu.cn`)
- When, where?
  5 lectures, 09:50-12:15, Room 5105, Teaching Building No. 5

Practical information

- Who?
  Marianna Girlando (`m.girlando@uva.nl`)
  Sisi Yang (`yangss23@mails.tsinghua.edu.cn`)
  Xin Li (`lixin24@mails.tsinghua.edu.cn`)
- When, where?
  5 lectures, 09:50-12:15, Room 5105, Teaching Building No. 5
- Evaluation
  3 homework (each due before the next lecture), 60% final grade
  1 take-home exam (due on Sunday 20 July, 23:59), 40% final grade

Practical information

- Who?
  Marianna Girlando (`m.girlando@uva.nl`)
  Sisi Yang (`yangss23@mails.tsinghua.edu.cn`)
  Xin Li (`lixin24@mails.tsinghua.edu.cn`)
- When, where?
  5 lectures, 09:50-12:15, Room 5105, Teaching Building No. 5
- Evaluation
  3 homework (each due before the next lecture), 60% final grade
  1 take-home exam (due on Sunday 20 July, 23:59), 40% final grade
- Material
  Annotated slides, uploaded daily on the course website

What is this course about?

Proof Theory of Modal Logic

What is this course about?

Proof Theory of Modal Logic

Proof theory is the discipline studying proofs as mathematical objects.

What is this course about?

Proof Theory of Modal Logic

Proof theory is the discipline studying **proofs** as **mathematical objects**.

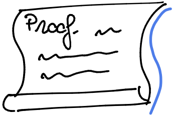
There are inf. many prime numbers $\rightsquigarrow$ formal language $\rightsquigarrow$ A

What is this course about?

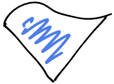
Proof Theory of Modal Logic

> Proof theory is the discipline studying proofs as mathematical objects.

There are inf. many prime numbers ⇝ formal language ⇝ A



⇝ proof system ⇝



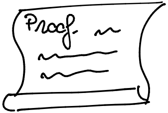
A

What is this course about?

Proof Theory of Modal Logic

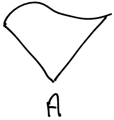
Proof theory is the discipline studying proofs as mathematical objects.

There are inf. many prime numbers ⇝ formal language ⇝ A



⇝ proof system ⇝

- axioms
- inference rules

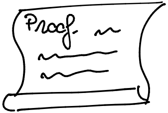


What is this course about?

Proof Theory of Modal Logic

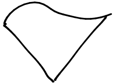
Proof theory is the discipline studying proofs as mathematical objects.

There are inf. many prime numbers $\rightsquigarrow$ formal language $\rightsquigarrow$ A



$\rightsquigarrow$ proof system $\rightsquigarrow$

- axioms
- inference rules



A

"*Ask three modal logicians what modal logic is, and you are likely to get at least three different answers.*" [Blackburn, de Rijke, Venema, 2001]

Modal languages are simple yet expressive languages for talking about relational structures.

Plan of the course

In this course, we will focus on sequent calculus, and explore various systems of sequent calculus for classical modal logics in the S5-cube.

Plan of the course

In this course, we will focus on sequent calculus, and explore various systems of sequent calculus for classical modal logics in the S5-cube.

- Lecture 1: Sequent Calculus and Modal Logic
- Lecture 2: Nested Sequents
- Lecture 3: Labelled Proof Systems
- Lecture 4: Semantic Completeness of Labelled Calculi
- Lecture 5: Beyond the S5-cube

Plan of the course

In this course, we will focus on sequent calculus, and explore various systems of sequent calculus for classical modal logics in the S5-cube.

- Lecture 1: Sequent Calculus and Modal Logic
- Lecture 2: Nested Sequents
- Lecture 3: Labelled Proof Systems
- Lecture 4: Semantic Completeness of Labelled Calculi
- Lecture 5: Beyond the S5-cube

Credits: This course is based on a course taught at ESSLLI 2024, which was prepared and taught in collaboration with Tiziano Dalmonte (Free University of Bozen-Bolzano, Italy).

This lecture: Sequent Calculus and Modal Logic

- Gentzen's sequent calculus
- G3-style sequent calculus
- G3-style sequent calculus for modal logics

Gentzen's sequent calculus

Sequent calculus

Introduced by Gerhard Gentzen in [Gentzen, 1935]

- As an auxiliary tool for natural deduction normalization
- Used to prove decidability of intuitionistic propositional logic
- [Gentzen, 1936]: proof of consistency of Peano Arithmetic

Sequent calculus

Introduced by Gerhard Gentzen in [Gentzen, 1935]

- As an auxiliary tool for natural deduction normalization
- Used to prove decidability of intuitionistic propositional logic
- [Gentzen, 1936]: proof of consistency of Peano Arithmetic

In sequent calculus, the basic components of proofs are not formulas (as in axiomatic systems or natural deduction), but sequents:

$$\Gamma \Rightarrow \Delta$$

for Γ, Δ are (possibly empty, finite) lists/sets/multisets of formulas.

Sequent calculus

Introduced by Gerhard Gentzen in [Gentzen, 1935]

- As an auxiliary tool for natural deduction normalization
- Used to prove decidability of intuitionistic propositional logic
- [Gentzen, 1936]: proof of consistency of Peano Arithmetic

In sequent calculus, the basic components of proofs are not formulas (as in axiomatic systems or natural deduction), but sequents:

$$\text{anteced.}\quad \Gamma \Rightarrow \Delta \quad \text{- consequent}$$

for Γ, Δ are (possibly empty, finite) lists/sets/multisets of formulas.

A sequent can be thought of as expressing consequence relation: at least one formula in Δ follows from the assumptions in Γ.

Classical propositional logic (CPL)

Atm set of propositional atoms, $p \in Atm$

$\neg A := A \to \bot$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A$$

Classical propositional logic (CPL)

Atm set of propositional atoms, $p \in Atm$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A$$

Hilbert-style axiom system $\quad \Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$

$$(A \wedge B) \to A$$

$$\bot \to A$$

$$\vdots$$

$$A \vee (A \to \bot)$$

$$\mathrm{mp}\ \frac{A \to B \quad A}{B}$$

Classical propositional logic (CPL)

Atm set of propositional atoms, $p \in Atm$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

Hilbert-style axiom system $\boxed{\Gamma \vdash_{\mathcal{H}cp} A}$

$$(A \wedge B) \rightarrow A$$

$$\bot \rightarrow A$$

$$\vdots$$

$$A \vee (A \rightarrow \bot)$$

$$\text{mp}\ \frac{A \rightarrow B \quad A}{B}$$

Semantics $\boxed{\Gamma \models_{cp} A}$

Propositional evaluation $\mathcal{A} : \underline{Atm} \longrightarrow \{0, 1\}$

$$\begin{array}{rcc} \mathcal{A} \models p & \textit{iff} & \mathcal{A}(p) = 1 \\ \mathcal{A} \models A \vee B & \textit{iff} & \mathcal{A} \models A \text{ or } \mathcal{A} \models B \\ & \vdots & \end{array}$$

Classical propositional logic (CPL)

Atm set of propositional atoms, $p \in Atm$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A$$

Hilbert-style axiom system $\quad \Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$

$$(A \wedge B) \to A$$
$$\bot \to A$$
$$\vdots$$
$$A \vee (A \to \bot)$$

$$\mathrm{mp}\ \frac{A \to B \quad A}{B}$$

Semantics $\quad \Gamma \models_{\mathrm{cp}} A$

Propositional evaluation $\mathcal{A} : Atm \longrightarrow \{0, 1\}$

$$\begin{array}{rcc} \mathcal{A} \models p & \textit{iff} & \mathcal{A}(p) = 1 \\ \mathcal{A} \models A \vee B & \textit{iff} & \mathcal{A} \models A \text{ or } \mathcal{A} \models B \\ & \vdots & \end{array}$$

Theorem. $\Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$ if and only if $\Gamma \models_{\mathrm{cp}} A$.

Gentzen's sequent calculus - **G1cp**

Basic Proof Theory

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge^i_L\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee^i_R\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

initial sequents

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

initial sequents

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

principal formula

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

initial sequents

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \perp_L\ \frac{}{\perp \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

active formulas

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

principal formula

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

initial sequents

logical rules

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

active formulas

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

principal formula

contexts

structural rules

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

cut formula

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

initial sequents

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_{\mathrm{L}}\ \frac{}{\bot \Rightarrow}$$

$$\wedge^i_{\mathrm{L}}\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_{\mathrm{R}}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_{\mathrm{L}}\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee^i_{\mathrm{R}}\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

active formulas

$$\to_{\mathrm{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_{\mathrm{R}}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

principal formula

contexts

$$\mathrm{wk_L}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{wk_R}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \mathrm{ctr_L}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{ctr_R}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\mathrm{ex_L}\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \mathrm{ex_R}\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \mathrm{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

cut formula

Derivations

A derivation, or proof, of $\Gamma \Rightarrow \Delta$ in **G1cp** is a finite tree whose nodes are labelled with sequents, and such that:

- The root of the tree is labelled with $\Gamma \Rightarrow \Delta$
- The leaves are labelled with initial sequents ($p \Rightarrow p$ or $\perp \Rightarrow$)
- Each internal node is obtained from its children by the application of a rule of **G1cp**

The height of a derivation is the length of its maximal branch, minus 1.

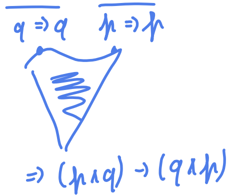
We write $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ if there is a derivation of $\Gamma \Rightarrow \Delta$ in **G1cp**.

Example

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}$$

$$\to_R\ \dfrac{\text{ctr}_L\ \dfrac{\wedge_L^2\ \dfrac{\text{ex}_L\ \dfrac{\wedge_L^1\ \dfrac{\wedge_R\ \dfrac{\text{wk}_L\ \dfrac{\text{init}\ \dfrac{}{q \Rightarrow q}}{p, q \Rightarrow q} \qquad \text{ex}_L\ \dfrac{\text{wk}_L\ \dfrac{\text{init}\ \dfrac{}{p \Rightarrow p}}{q, p \Rightarrow p}}{p, q \Rightarrow p}}{p, q \Rightarrow q \wedge p}}{p \wedge q, q \Rightarrow q \wedge p}}{q, p \wedge q \Rightarrow q \wedge p}}{p \wedge q, p \wedge q \Rightarrow q \wedge p}}{p \wedge q \Rightarrow q \wedge p}}{\Rightarrow (p \wedge q) \to (q \wedge p)}$$



Soundness and completeness

$$i(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma) \to (\bigvee \Delta)$$

Theorem (Soundness). If $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\mathrm{cp}} i(\Gamma \Rightarrow \Delta)$.

Proof sketch. By showing that the initial sequents are derivable in the Hilbert system $\mathcal{H}$cp, and the rules of **G1cp** preserve derivability in $\mathcal{H}$cp.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$ then $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$.

Proof sketch. By deriving the axioms and simulating the rules of $\mathcal{H}$cp.

Soundness and completeness

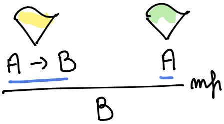
$$i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

Theorem (Soundness). If $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\mathrm{cp}} i(\Gamma \Rightarrow \Delta)$.

Proof sketch. By showing that the initial sequents are derivable in the Hilbert system $\mathcal{H}$cp, and the rules of **G1cp** preserve derivability in $\mathcal{H}$cp.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$ then $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$.

Proof sketch. By deriving the axioms and simulating the rules of $\mathcal{H}$cp.



Soundness and completeness

$i(p, q \Rightarrow r, s) = (p \wedge q) \to (r \vee s)$

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}$$

$$i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

Theorem (Soundness). If $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\mathrm{cp}} i(\Gamma \Rightarrow \Delta)$.

Proof sketch. By showing that the initial sequents are derivable in the Hilbert system $\mathcal{H}$cp, and the rules of **G1cp** preserve derivability in $\mathcal{H}$cp.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$ then $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$.

Proof sketch. By deriving the axioms and simulating the rules of $\mathcal{H}$cp.

$$\dfrac{A \to B \qquad A}{B}\,\text{mp} \quad \rightsquigarrow \quad \dfrac{\overset{\text{I.H.}}{\Rightarrow A} \qquad \dfrac{\overset{\text{I.H.}}{\Rightarrow A \to B} \qquad \dfrac{A \Rightarrow B, A \qquad B, A \Rightarrow B}{A \to B, A \Rightarrow B}\,{\to}_L}{A \Rightarrow B}\,\text{cut}}{\Rightarrow B}\,\text{cut}$$

Derivable, admissible, eliminable rule

$R \notin SC$ $\qquad$ $n \leq 2$ $\qquad$ sequents

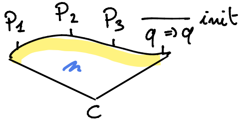
SC sequent calculus (set of rules) and R rule ($n \geq 0$): $\quad R\,\dfrac{P_1 \quad \ldots \quad P_n}{C}$

R is derivable: There is a derivation of C in SC such that every leaf of the tree is labelled with an initial sequent or a premiss P_i of R

Derivable, admissible, eliminable rule

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \ldots \quad P_n}{C}$$

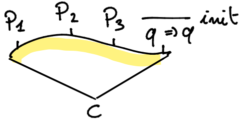
R is derivable: There is a derivation of C in SC such that every leaf of the tree is labelled with an initial sequent or a premiss P_i of R



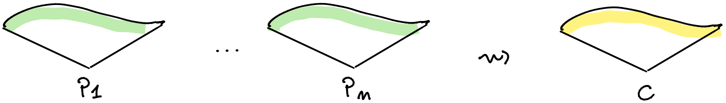
Derivable, admissible, eliminable rule

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is derivable: There is a derivation of C in SC such that every leaf of the tree is labelled with an initial sequent or a premiss P_i of R



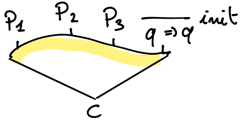
R is admissible: If each premiss P_i is derivable in SC, then C is also derivable in SC.



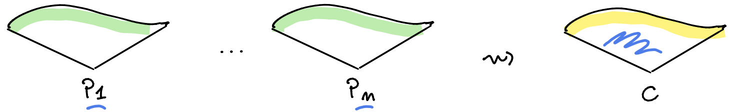
Derivable, admissible, eliminable rule

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is derivable: There is a derivation of C in SC such that every leaf of the tree is labelled with an initial sequent or a premiss P_i of R

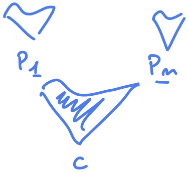


R is admissible: If each premiss P_i is derivable in SC, then C is also derivable in SC.

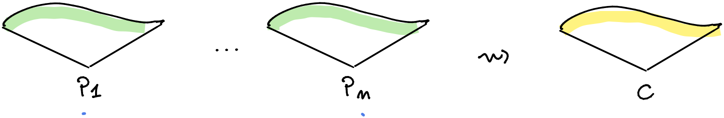


R is eliminable from $\text{SC} \cup \{R\}$: Every derivation in $\text{SC} \cup \{R\}$ can be transformed in a derivation in SC

Derivable, admissible, eliminable rule



- If R is derivable in SC then R is admissible in SC



- If R is eliminable in SC $\cup \{R\}$ then R is admissible in SC

Cut elimination and consistency

R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

Cut elimination and consistency

R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

$$\to_L \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{cut} \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut elimination and consistency

R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

$$\rightarrow_L \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{cut} \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- cut is the only non-analytic rule of **G1cp**

Cut elimination and consistency

R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

$$\rightarrow_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- cut is the only non-analytic rule of **G1cp**
- If we don't have cut, we cannot derive $\Rightarrow \bot$

$$\frac{\begin{matrix} \vdots & & \vdots \\ \Rightarrow \bot, p & & p \Rightarrow \end{matrix}}{\Rightarrow \bot}\ \text{cut}$$

Cut elimination and consistency

R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

$$\rightarrow_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- cut is the only non-analytic rule of **G1cp**
- If we don't have cut, we cannot derive $\Rightarrow \bot$
- If we don't have cut, we can prove that CPL is consistent

Cut elimination and consistency

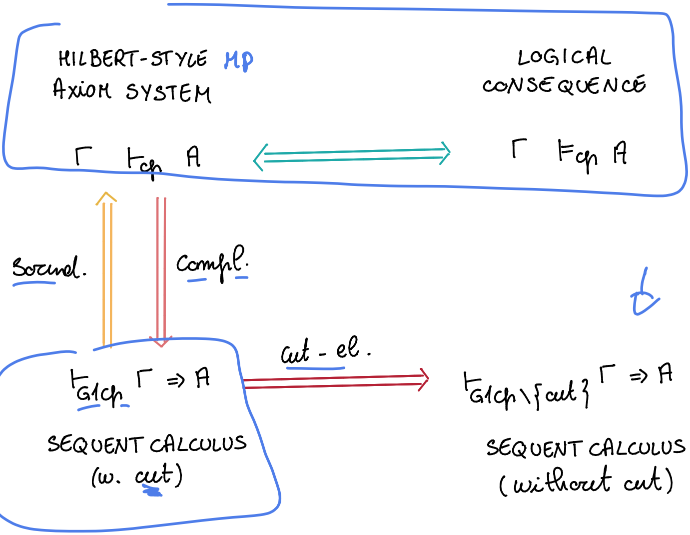
R is analytic: If every formula occurring in the premisses of R is a subformula of some formula occurring in the conclusion.

$$\to_L \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{cut}\, \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- cut is the only non-analytic rule of **G1cp**
- If we don't have cut, we cannot derive $\Rightarrow \bot$
- If we don't have cut, we can prove that CPL is consistent

Theorem (Cut). Every derivation in **G1cp** ~~∪{cut}~~ can be transformed into a derivation in **G1cp**.$\setminus \{cut\}$

Roadmap


HILBERT-STYLE MP
AXIOM SYSTEM
$\Gamma \vdash_{cp} A$
LOGICAL CONSEQUENCE
$\Gamma \vDash_{cp} A$
Sound.
Compl.
cut-el.
$\vdash_{G1cp} \Gamma \Rightarrow A$
SEQUENT CALCULUS
(w. cut)
$\vdash_{G1cp \setminus \{cut\}} \Gamma \Rightarrow A$
SEQUENT CALCULUS
(without cut)

G3-style sequent calculus

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Removing the structural rules from **G1cp**

$p, q, q \neq p, q = q, p$

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad\qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge^i_L\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad\qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \vee^i_R\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad\qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta} \qquad \text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi} \qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

multisets : sets where the multiplicity (# of occurences) matters, but the order does not matter

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad\qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad\qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad\qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\cancel{\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}} \qquad \cancel{\text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}} \qquad \cancel{\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}$$

we will prove cut-admissibility

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\text{wk}_L\ \cancel{\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \text{wk}_R\ \cancel{\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \cancel{\frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}} \qquad \text{ex}_R\ \cancel{\frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}} \qquad \text{cut}\ \cancel{\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}$$

we will prove cut-admissibility

Removing the structural rules from **G1cp**

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad\qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad\qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\rightarrow_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad\qquad \rightarrow_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$\text{wk}_L\ \cancel{\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \text{wk}_R\ \cancel{\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}} \qquad \text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\text{ex}_L\ \cancel{\frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}} \qquad \text{ex}_R\ \cancel{\frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}} \qquad \text{cut}\ \cancel{\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}$$

we will prove cut-admissibility

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad\qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad\qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \vee_R^i\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ i \in \{1,2\}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad\qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\boxed{\cancel{\text{wk}_L\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \cancel{\text{wk}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}}} \qquad \boxed{\cancel{\text{ctr}_L\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \cancel{\text{ctr}_R\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}}}$$

$$\boxed{\cancel{\text{ex}_L\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}} \qquad \cancel{\text{ex}_R\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}}} \qquad \boxed{\cancel{\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}}$$

we will prove cut-admissibility

Removing the structural rules from **G1cp**

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot\text{L}\ \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

$$\wedge^i_\text{L}\ \frac{A_1, A_2, \cancel{A_i}, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ \cancel{i \in \{1,2\}} \qquad \wedge\text{R}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee\text{L}\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee^i_\text{R}\ \frac{\Gamma \Rightarrow \Delta, \cancel{A_i}, A_1, A_2}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ \cancel{i \in \{1,2\}}$$

$$\to\text{L}\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to\text{R}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\cancel{\text{wk}_\text{L}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \cancel{\text{wk}_\text{R}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}} \qquad \cancel{\text{ctr}_\text{L}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}} \qquad \cancel{\text{ctr}_\text{R}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}}$$

$$\cancel{\text{ex}_\text{L}\ \frac{\Gamma, A, B, \Sigma \Rightarrow \Delta}{\Gamma, B, A, \Sigma \Rightarrow \Delta}} \qquad \cancel{\text{ex}_\text{R}\ \frac{\Gamma \Rightarrow \Delta, A, B, \Pi}{\Gamma \Rightarrow \Delta, B, A, \Pi}} \qquad \cancel{\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}$$

we will prove cut-admissibility

Sequent calculus **G3cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ multisets of formulas

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad \bot_L\ \frac{}{\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_L\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Example

G1cp

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$q \Rightarrow q$}
\LeftLabel{wk_L}
\UnaryInfC{$p, q \Rightarrow q$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$p \Rightarrow p$}
\LeftLabel{wk_L}
\UnaryInfC{$q, p \Rightarrow p$}
\LeftLabel{ex_L}
\UnaryInfC{$p, q \Rightarrow p$}
\LeftLabel{$\wedge_\text{R}$}
\BinaryInfC{$p, q \Rightarrow q \wedge p$}
\LeftLabel{$\wedge_\text{L}^1$}
\UnaryInfC{$p \wedge q, q \Rightarrow q \wedge p$}
\LeftLabel{ex_L}
\UnaryInfC{$q, p \wedge q \Rightarrow q \wedge p$}
\LeftLabel{$\wedge_\text{L}^2$}
\UnaryInfC{$p \wedge q, p \wedge q \Rightarrow q \wedge p$}
\LeftLabel{ctr_L}
\UnaryInfC{$p \wedge q \Rightarrow q \wedge p$}
\LeftLabel{$\rightarrow_\text{R}$}
\UnaryInfC{$\Rightarrow (p \wedge q) \rightarrow (q \wedge p)$}
\end{prooftree}
$$

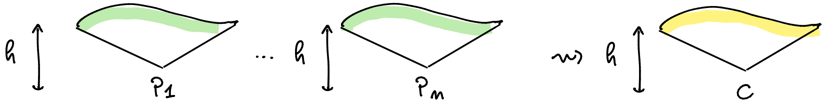
G3cp

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$p, q \Rightarrow q$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$p, q \Rightarrow p$}
\RightLabel{$\wedge R$}
\BinaryInfC{$p, q \Rightarrow q \wedge p$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \wedge q \Rightarrow q \wedge p$}
\RightLabel{$R \rightarrow$}
\UnaryInfC{$\Rightarrow (p \wedge q) \rightarrow (q \wedge p)$}
\end{prooftree}
$$

Structural properties of **G3cp**

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $${}_{R}\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is height-preserving admissible (hp-admissible): If each premiss P_i is derivable in SC with derivations of height of at most h, then C is also derivable in SC with a derivation of height of at most h.



Lemma (Weakening). The weakening rules are hp-admissible in **G3cp**.

Lemma (Contraction). The contraction rules are hp-admissible in **G3cp**.

Theorem (Cut). The cut rule is admissible in **G3cp**.

Soundness and completeness

$$i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

Theorem (Soundness). If $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\text{cp}} i(\Gamma \Rightarrow \Delta)$.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\text{cp}} A$ then $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow A$.

Roadmap

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\Gamma \vdash_{cp} A \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow, shift left, "\text{compl. (via cut-adm)}"] & \Gamma \vDash_{cp} A \\
& \vdash_{G3cp} \Gamma \Rightarrow A \arrow[ul, Rightarrow, shift left, "\text{sound.}"]
\end{tikzcd}$$

SEQUENT CALCULUS (without cut)

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $\quad R\ \dfrac{P_1 \quad \dots \quad P_n}{C}$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

Example

not inv.!

$$\wedge_L^1\ \dfrac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad\qquad \wedge_L\ \dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

$$\wedge_L^1\ \dfrac{\overset{\times}{q \Rightarrow p}}{p \wedge q \Rightarrow p} \qquad\qquad \wedge_L^2\ \dfrac{\text{init}\ \dfrac{}{p \Rightarrow p}}{p \wedge q \Rightarrow p}$$

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\ \frac{P_1 \quad \ldots \quad P_n}{C}$$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

Example

$$\wedge_L^1\ \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

Lemma (Invertibility). All the rules of **G3cp** are hp-invertible.

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

Example

$$\wedge_L^1\,\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L\,\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

Lemma (Invertibility). All the rules of **G3cp** are hp-invertible.

☞ Why is invertibility important?

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

Example

$$\wedge_L^1\,\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad\qquad \wedge_L\,\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

Lemma (Invertibility). All the rules of **G3cp** are hp-invertible.

☞ Why is invertibility important? Interlude: decision procedures

Made with Xodo PDF Reader and Editor

Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

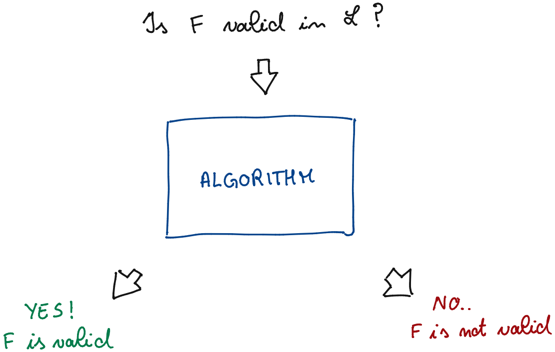
Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

$$\Downarrow$$

$\boxed{\text{ALGORITHM}}$

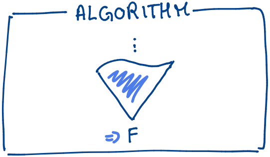
Decision procedures and proof theory



Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

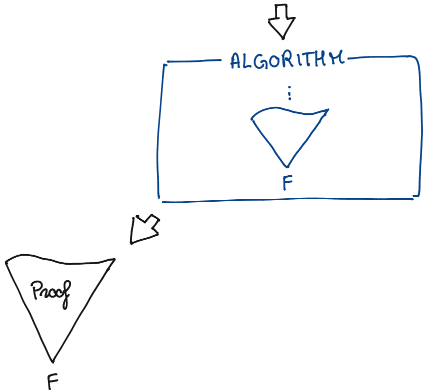
$\Downarrow$



P proof system for $\mathcal{L}$

Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

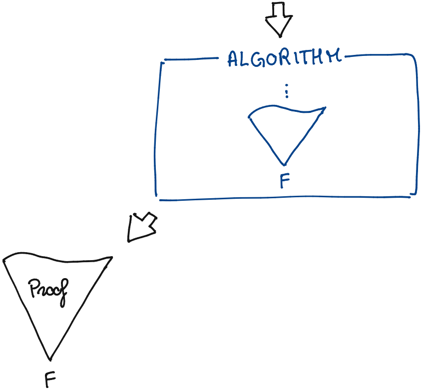


P proof system for $\mathcal{L}$

Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

P proof system for $\mathcal{L}$

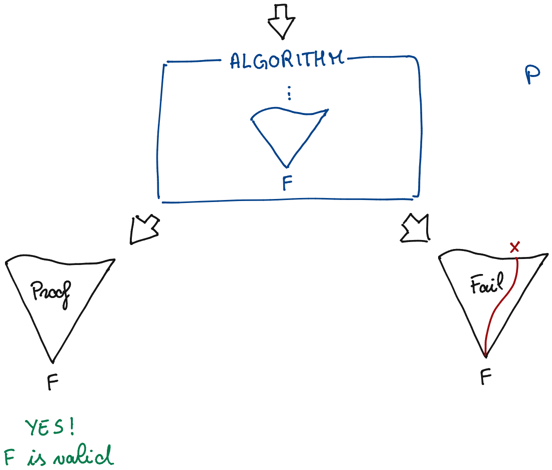


YES!
F is valid

Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

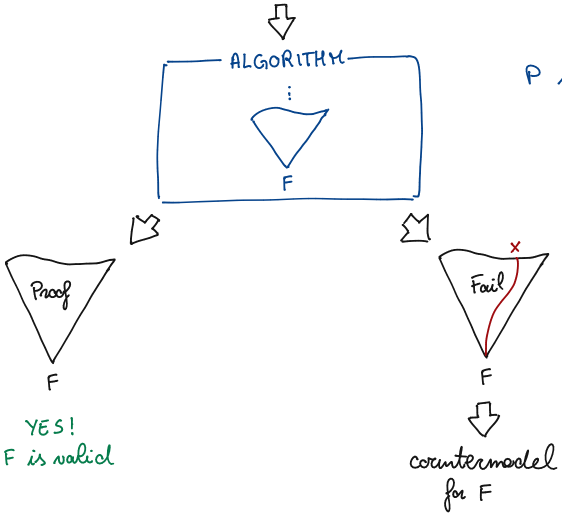
P proof system for $\mathcal{L}$



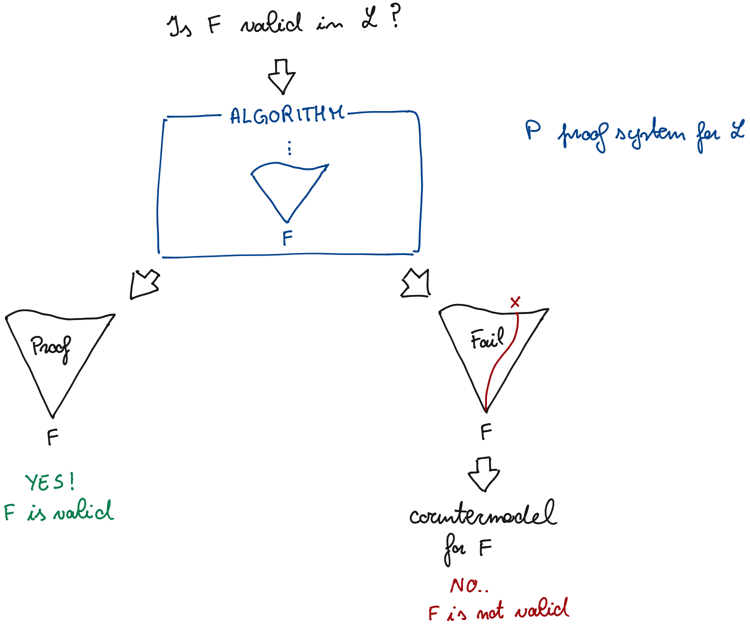
Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

P proof system for $\mathcal{L}$



Decision procedures and proof theory



Desirable properties for proof search

- **Termination** of backward proof search

  $\rightsquigarrow$ guaranteed by analyticity, and by the fact that the rules reduce the complexity of sequents

- Decision procedure by a single proof-search tree: it suffices to construct *one* derivation tree to check for derivability

  $\rightsquigarrow$ guaranteed by invertibility of all the rules

- Countermodel construction from (a leaf of) a failed branch

  $\rightsquigarrow$ read the formulas in the antecedent as “true”, and those in the consequent as “false”

Summing up

	fml. interpr.	invertible rules	analyti- city	termination proof search	counterm. constr.	modu- larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a

G3-style sequent calculus for modal logic ?

The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

dual $\quad \Diamond A \leftrightarrow \neg\Box\neg A$

k $\quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

$$\text{nec}\ \frac{A}{\Box A}$$

The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

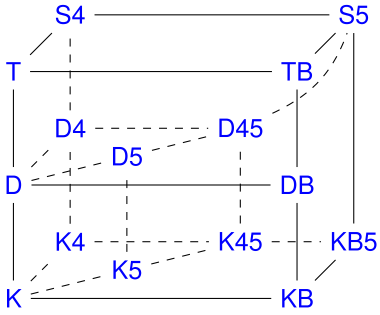
dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$
	$\text{nec}\ \dfrac{A}{\Box A}$
d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$

The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \to B) \to (\Box A \to \Box B)$
	$\text{nec}\ \dfrac{A}{\Box A}$
d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box\Diamond A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$

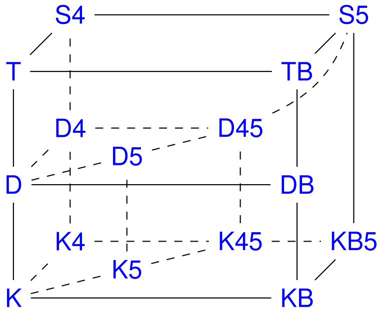


The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

dual	$\Diamond A \leftrightarrow \neg\Box\neg A$
k	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
	nec $\dfrac{A}{\Box A}$
d	$\Box A \rightarrow \Diamond A$
t	$\Box A \rightarrow A$
b	$A \rightarrow \Box\Diamond A$
4	$\Box A \rightarrow \Box\Box A$
5	$\Diamond A \rightarrow \Box\Diamond A$



$\Gamma \vdash A \rightsquigarrow$ A is derivable from Γ in $\mathcal{H}$K

For $X \subseteq \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, $\Gamma \vdash_X A \rightsquigarrow$ A is derivable from Γ in $\mathcal{H}\mathrm{K} \cup X$

Kripke models

$$\mathcal{M} = \langle W, R, v \rangle$$

- W non-empty set of elements (*worlds*)
- R binary relation on W (*accessibility relation*)
- v *valuation function* $W \longrightarrow \mathcal{P}(Atm)$

Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box \Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \to xRz)$
5	$\Diamond A \to \Box \Diamond A$	Euclideaness	$\forall x \forall y \forall z ((xRy \wedge xRz) \to yRz)$

Notation. For $X \subseteq \{d, t, b, 4, 5\}$ We denote by $\mathcal{X}$ the class of all models satisfying all conditions corresponding to the axioms of the logic X

Soundness and completeness

Satisfiability $\underline{\mathcal{M}, w} \Vdash A$

$$\begin{array}{rcl} \mathcal{M}, w \Vdash p & \textit{iff} & p \in v(w) \\ & (..) & \\ \mathcal{M}, w \Vdash \underline{\Box A} & \textit{iff} & \text{for all } u \text{ s.t. } wRu, u \Vdash A \\ \mathcal{M}, w \Vdash \Diamond A & \textit{iff} & \text{there exists } u \text{ s.t. } wRu \text{ and } u \Vdash A \end{array}$$

Validity in a model $\mathcal{M} \models A$ *iff* for all $w \in \mathcal{M}$, $\mathcal{M}, w \Vdash A$

Validity in a class of models $\underline{\models_{\mathcal{X}} A}$ *iff* for all $\mathcal{M} \in \mathcal{X}$, $\mathcal{M} \models A$

Logical consequence

$\underline{\Gamma \models_{\mathcal{X}} A}$ *iff* for all $\mathcal{M} \in \mathcal{X}$, for all $w \in \mathcal{M}$,
if $\mathcal{M}, w \Vdash B$ for all $B \in \Gamma$, *then* $\mathcal{M}, w \Vdash A$

Theorem. $\underline{\Gamma \vdash_{\mathrm{X}} A}$ if and only if $\underline{\Gamma \models_{\mathcal{X}} A}$ [Blackburn de Rijke, Venema, 2001]

Sequent calculi for (some) modal logics

For simplicity, we define the rules for the $\Box$-only fragment.

Some references: [Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

Sequent calculi for (some) modal logics

For simplicity, we define the rules for the □-only fragment.

Some references: [Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

$$\mathbf{G3K} \; = \; \mathbf{G3cp} \; + \; {}_{\mathrm{k}}\frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \text{ for } n \geq 0$$

α

Sequent calculi for (some) modal logics

For simplicity, we define the rules for the $\Box$-only fragment.
Some references: [Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

$$\mathbf{G3K} \;=\; \mathbf{G3cp} \;+\; \text{k}\,\frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \underline{\Box B_1, \ldots, \Box B_n} \Rightarrow \Box A, \Delta}\ \text{for } n \geq 0$$

Notation. Given $\Sigma = B_1, \ldots, B_n$ (for $n \geq 0$), let $\Box\Sigma = \Box B_1, \ldots, \Box B_n$.
The rule k can be written as

$$\text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Sequent calculi for (some) modal logics

For simplicity, we define the rules for the $\Box$-only fragment.
Some references: [Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

$$\mathbf{G3K} \;=\; \mathbf{G3cp} \;+\; \mathrm{k}\,\frac{B_1,\ldots,B_n \Rightarrow A}{\Gamma, \Box B_1,\ldots,\Box B_n \Rightarrow \Box A, \Delta}\ \text{for } n \geq 0$$

Notation. Given $\Sigma = B_1,\ldots,B_n$ (for $n \geq 0$), let $\Box\Sigma = \Box B_1,\ldots,\Box B_n$.
The rule k can be written as

$$\mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

$$\mathrm{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad \mathrm{4}\,\frac{\Box\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad \mathrm{45}\,\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

- Sequent calculus for T: **G3K** $\cup$ {t}
- Sequent calculus for S4: **G3cp** $\cup$ {4, t}
- Sequent calculus for S5: **G3cp** $\cup$ {45, t}

Sequent calculus for K

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; {}_{\mathsf{k}}\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Structural properties of **G3cp**

- Weakening and contraction are hp-admissible
- All propositional rules are hp-invertible (but not the rule k)
- Cut is admissible

Theorem (Soundness). If $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow \Delta$ then $\vdash i(\Gamma \Rightarrow \Delta)$.

Theorem (Completeness). If $\Gamma \vdash A$ then $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow A$.

Invertibility

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Rule k is not invertible

Invertibility

$$\textbf{G3K} \quad = \quad \textbf{G3cp} \; + \; \mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Rule k is not invertible

Consequently:

- One failed proof is not sufficient to ensure non-derivability
- Hence, in particular, it does not provide a countermodel
- Backward proof-search in **G3K** requires backtracking

Example

Is the sequent $\Box(p \vee q) \Rightarrow \Box p \vee \Box q$ derivable in **G3K**?

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{p \Rightarrow p}\,\text{init} \qquad \boxed{\begin{array}{c} \times \\ q \Rightarrow p \end{array}} \Leftarrow 2}{p \vee q \Rightarrow p}\,\vee_L
}{}\,K
\qquad
\dfrac{
\dfrac{\boxed{\begin{array}{c} \times \\ p \Rightarrow q \end{array}} \Leftarrow 3 \qquad \dfrac{}{q \Rightarrow q}\,\text{init}}{p \vee q \Rightarrow q}\,\vee_L
}{}\,K
}{
1 \Rightarrow \quad \dfrac{\Box(p \vee q) \Rightarrow \Box p, \Box q}{\Box(p \vee q) \Rightarrow \Box p \vee \Box q}\,\vee_R
}
$$

$$
\begin{array}{l}
\Vdash \Box(p \vee q) \\
\nVdash \Box p \\
\nVdash \Box q
\end{array}
\quad 1 \quad
\begin{array}{l}
\nearrow \; 2 \Vdash q,\ 2 \nVdash p \\[2em]
\searrow \; 3 \Vdash p,\ 3 \nVdash q
\end{array}
$$

Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
G3S5	yes	no	<u>no</u>			

G3-style sequent calculus

enrich the <u>structure</u> of sequents
nested sequents

enrich the <u>language</u> of sequents
labelled calculus

End of content for today's lecture!

Questions?

1. Show that the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and the rule $\dfrac{\Rightarrow A}{\Rightarrow \Box A}$ are derivable in **G3K**.
2. We wish to show that **G3T** is not contraction-free complete [Goré, 1999]. Sequent calculus **G3T** adds the following two rules to **G3cp**:

$$\mathrm{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad \mathrm{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

We consider formula $\Box(p \wedge (\Box p \to \bot) \to \bot$, which is valid in T. Show that:

   a) The sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ is derivable in $\mathbf{G3T} \cup \{\mathrm{ctr_L}, \mathrm{ctr_R}\}$
   b) The sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ is not derivable in **G3T**
   c) If we substitute rule t with the following rule t′ in **G3T**, then sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ becomes derivable (without contraction):

$$\mathrm{t}'\,\frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

3. Next, we wish to show that **G3S5** is not cut-free complete. Sequent calculus **G3S5** adds the following rules to **G3cp**:

$$\mathrm{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad 45\,\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

We consider formula $p \vee \Box(\Box p \to \bot)$, which is valid in S5. Show that:

   a) The sequent $p \vee \Box(\Box p \to \bot)$ is derivable in $\mathbf{G3S5} \cup \{\mathrm{cut}\}$
   b) The sequent $p \vee \Box(\Box p \to \bot)$ is not derivable in **G3S5**