Proof Theory of Modal Logic

Lecture 1
Sequent Calculus and Modal Logic

Marianna Girlando

ILLC, Universtiy of Amsterdam

5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Practical information

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- Who?
  Marianna Girlando (m.girlando@uva.nl)
  Sisi Yang (yangss23@mails.tsinghua.edu.cn)
  Xin Li (lixin24@mails.tsinghua.edu.cn)

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  Sisi Yang (yangss23@mails.tsinghua.edu.cn)
  Xin Li (lixin24@mails.tsinghua.edu.cn)
- When, where?
  5 lectures, 09:50-12:15, Room 5105, Teaching Building No. 5

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- When, where?
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- Evaluation
  3 homework (each due before the next lecture), 60% final grade
  1 take-home exam (due on Sunday 20 July, 23:59), 40% final grade

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- Material
  Annotated slides, uploaded daily on the course website

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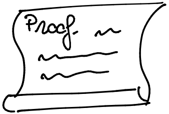
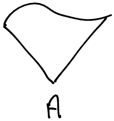


⇝ proof system ⇝



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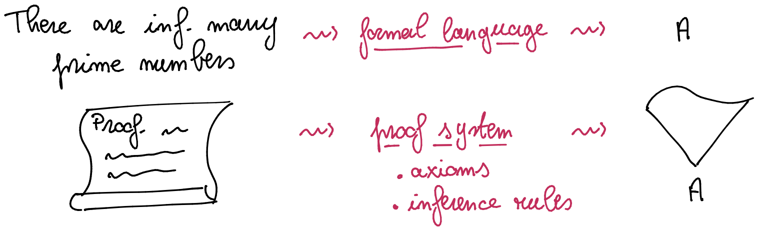

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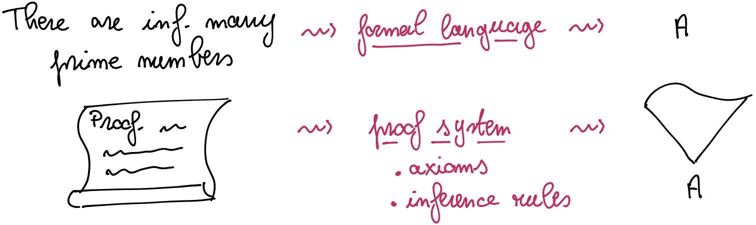

"*Ask three modal logicians what modal logic is, and you are likely to get at least three different answers.*" [Blackburn, de Rijke, Venema, 2001]

Modal languages are simple yet expressive languages for talking about **relational structures**.

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- Lecture 3: Labelled Proof Systems
- Lecture 4: Semantic Completeness of Labelled Calculi
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Credits: This course is based on a course taught at ESSLLI 2024, which was prepared and taught in collaboration with Tiziano Dalmonte (Free University of Bozen-Bolzano, Italy).

This lecture: Sequent Calculus and Modal Logic

- Gentzen's sequent calculus
- G3-style sequent calculus
- G3-style sequent calculus for modal logics

Gentzen's sequent calculus

Introduced by Gerhard Gentzen in [Gentzen, 1935]

- As an auxiliary tool for natural deduction normalization
- Used to prove decidability of intuitionistic propositional logic
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In sequent calculus, the basic components of proofs are not formulas (as in axiomatic systems or natural deduction), but sequents:

$$\Gamma \Rightarrow \Delta$$

for Γ, Δ are (possibly empty, finite) lists/sets/multisets of formulas.

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A sequent can be thought of as expressing consequence relation: at least one formula in Δ follows from the assumptions in Γ.

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Atm set of propositional atoms, $p \in Atm$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A$$

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Hilbert-style axiom system $\quad \Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$

$$(A \wedge B) \rightarrow A$$

$$\bot \rightarrow A$$

$$\vdots$$

$$A \vee (A \rightarrow \bot)$$

$$\text{mp}\ \frac{A \rightarrow B \quad A}{B}$$

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Semantics $\Gamma \models_{\text{cp}} A$

Propositional evaluation $\mathcal{A} : Atm \longrightarrow \{0, 1\}$

$$\begin{array}{rcl} \mathcal{A} \models p & \textit{iff} & \mathcal{A}(p) = 1 \\ \mathcal{A} \models A \vee B & \textit{iff} & \mathcal{A} \models A \text{ or } \mathcal{A} \models B \\ & \vdots & \end{array}$$

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Theorem. $\Gamma \vdash_{\mathcal{H}cp} A$ if and only if $\Gamma \models_{cp} A$.

Gentzen's sequent calculus - **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

Rules of **G1cp**:

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_{\text{L}}\ \frac{}{\bot \Rightarrow}$$

$$\wedge^i_{\text{L}}\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_{\text{R}}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

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cut formula

Derivations

A derivation, or proof, of $\Gamma \Rightarrow \Delta$ in **G1cp** is a finite tree whose nodes are labelled with sequents, and such that:

- The root of the tree is labelled with $\Gamma \Rightarrow \Delta$
- The leaves are labelled with initial sequents ($p \Rightarrow p$ or $\bot \Rightarrow$)
- Each internal node is obtained from its children by the application of a rule of **G1cp**

The height of a derivation is the length of its maximal branch, minus 1.

We write $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ if there is a derivation of $\Gamma \Rightarrow \Delta$ in **G1cp**.

Example

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \wedge^i_{\mathsf{L}}\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

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Soundness and completeness

$$i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$$

Theorem (Soundness). If $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\text{cp}} i(\Gamma \Rightarrow \Delta)$.

Proof sketch. By showing that the initial sequents are derivable in the Hilbert system $\mathcal{H}$cp, and the rules of **G1cp** preserve derivability in $\mathcal{H}$cp.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\text{cp}} A$ then $\vdash_{\mathbf{G1cp}} \Gamma \Rightarrow A$.

Proof sketch. By deriving the axioms and simulating the rules of $\mathcal{H}$cp.

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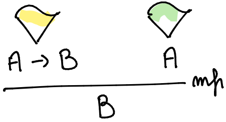

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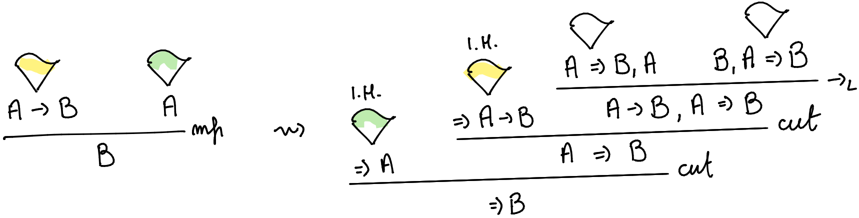

Derivable, admissible, eliminable rule

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \ldots \quad P_n}{C}$$

R is derivable: There is a derivation of C in SC such that every leaf of the tree is labelled with an initial sequent or a premiss P_i of R

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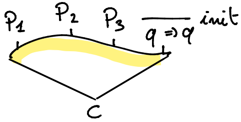

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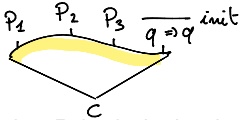
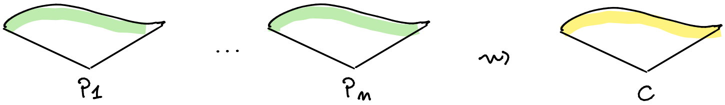


R is admissible: If each premiss P_i is derivable in SC, then C is also derivable in SC.



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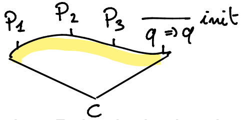
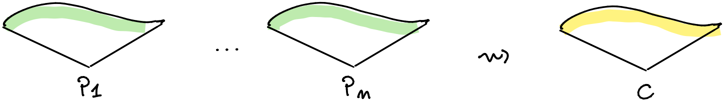


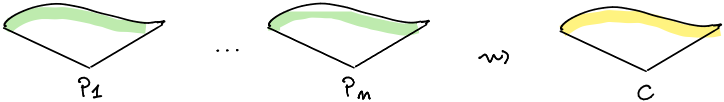
R is admissible: If each premiss P_i is derivable in SC, then C is also derivable in SC.



R is eliminable from SC $\cup \{R\}$: Every derivation in SC $\cup \{R\}$ can be transformed in a derivation in SC

Derivable, admissible, eliminable rule

- If R is derivable in SC then R is admissible in SC



- If R is eliminable in SC $\cup \{R\}$ then R is admissible in SC

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Cut elimination and consistency

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- cut is the only non-analytic rule of **G1cp**

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Cut elimination and consistency

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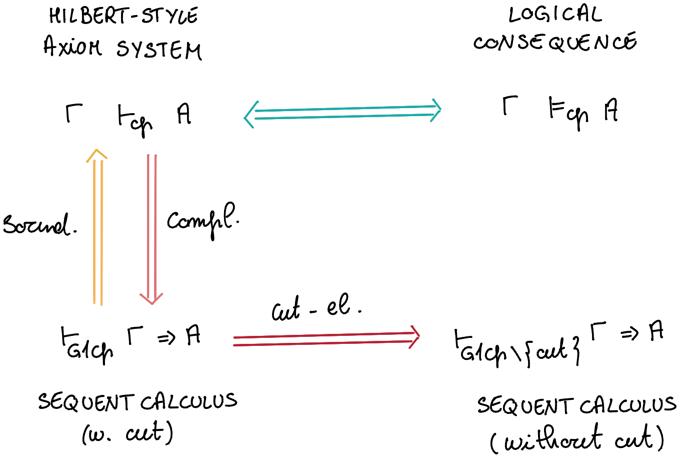
> Theorem (Cut). Every derivation in **G1cp** $\cup$ {cut} can be transformed into a derivation in **G1cp**.

Roadmap


HILBERT-STYLE AXIOM SYSTEM
LOGICAL CONSEQUENCE
$\Gamma \vdash_{cp} A$
$\Gamma \models_{cp} A$
Sound.
Compl.
cut-el.
$\vdash_{G1cp} \Gamma \Rightarrow A$
$\vdash_{G1cp \setminus \{cut\}} \Gamma \Rightarrow A$
SEQUENT CALCULUS (w. cut)
SEQUENT CALCULUS (without cut)

G3-style sequent calculus

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ lists of formulas

$$\frac{}{p \Rightarrow p}\,\text{init} \qquad \frac{}{\bot \Rightarrow}\,\bot_\mathrm{L}$$

$$\wedge^i_\mathrm{L}\,\frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\; i \in \{1,2\} \qquad \wedge_\mathrm{R}\,\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_\mathrm{L}\,\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee^i_\mathrm{R}\,\frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\; i \in \{1,2\}$$

$$\to_\mathrm{L}\,\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_\mathrm{R}\,\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\mathrm{wk_L}\,\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{wk_R}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \mathrm{ctr_L}\,\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathrm{ctr_R}\,\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

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Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

multisets: sets where the multiplicity (# of occurrences) matters, but the order does not matter

$$\text{init}\ \frac{}{p \Rightarrow p} \qquad \bot_L\ \frac{}{\bot \Rightarrow}$$

$$\wedge_L^i\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ i \in \{1,2\} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

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(The exchange rules ex_L and ex_R are circled and crossed out by hand.)

Removing the structural rules from **G1cp**

multisets : sets where the multiplicity (# of occurrences) matters, but the order does not matter

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(ex$_L$, ex$_R$ and cut are crossed out.)

we will prove cut-admissibility

Removing the structural rules from **G1cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ ~~lists~~ multisets of formulas

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(wk, ex and cut rules crossed out)

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we will prove cut-admissibility

Sequent calculus **G3cp**

Sequent: $\Gamma \Rightarrow \Delta$, for Γ and Δ multisets of formulas

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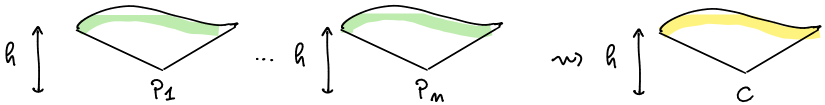
Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,\text{init}}{p, q \Rightarrow q}\,\text{wk}_\text{L} \qquad \dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,\text{init}}{q, p \Rightarrow p}\,\text{wk}_\text{L}}{p, q \Rightarrow p}\,\text{ex}_\text{L}}{p, q \Rightarrow q \wedge p}\,\wedge_\text{R}}{p \wedge q, q \Rightarrow q \wedge p}\,\wedge_\text{L}^1}{q, p \wedge q \Rightarrow q \wedge p}\,\text{ex}_\text{L}}{p \wedge q, p \wedge q \Rightarrow q \wedge p}\,\wedge_\text{L}^2}{p \wedge q \Rightarrow q \wedge p}\,\text{ctr}_\text{L}}{\Rightarrow (p \wedge q) \to (q \wedge p)}\,\to_\text{R}
$$

Structural properties of **G3cp**

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\ \frac{P_1 \quad \dots \quad P_n}{C}$$

R is height-preserving admissible (hp-admissible): If each premiss P_i is derivable in SC with derivations of height of at most h, then C is also derivable in SC with a derivation of height of at most h.



Lemma (Weakening). The weakening rules are hp-admissible in **G3cp**.

Lemma (Contraction). The contraction rules are hp-admissible in **G3cp**.

Theorem (Cut). The cut rule is admissible in **G3cp**.

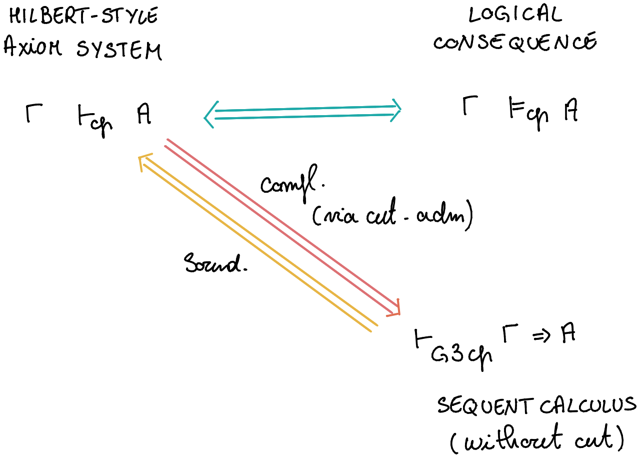
Soundness and completeness

$$i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

Theorem (Soundness). If $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$ then $\vdash_{\mathcal{H}\mathrm{cp}} i(\Gamma \Rightarrow \Delta)$.

Theorem (Completeness). If $\Gamma \vdash_{\mathcal{H}\mathrm{cp}} A$ then $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow A$.

Roadmap


HILBERT-STYLE
AXIOM SYSTEM
$\Gamma \vdash_{cp} A$
LOGICAL
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$\Gamma \models_{cp} A$
Compl.
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$\vdash_{G3cp} \Gamma \Rightarrow A$
SEQUENT CALCULUS
(without cut)

Invertibility

SC sequent calculus (set of rules) and R rule ($n \geq 0$): $$R\,\frac{P_1 \quad \dots \quad P_n}{C}$$

R is (height-preserving) invertible: If C is derivable in SC (with a derivation of height at most h), then every premiss P_i is derivable in SC (with a derivation of height at most h).

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Example

$$\wedge_L^1 \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_L \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

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Lemma (Invertibility). All the rules of **G3cp** are hp-invertible.

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☞ Why is invertibility important?

Invertibility

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Lemma (Invertibility). All the rules of **G3cp** are hp-invertible.

☞ Why is invertibility important? Interlude: decision procedures

Decision procedures and proof theory

Is F valid in $\mathcal{L}$?

Decision procedures and proof theory

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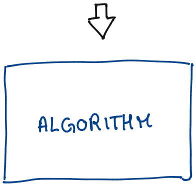
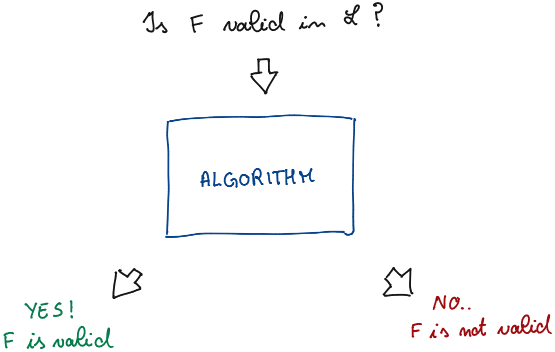
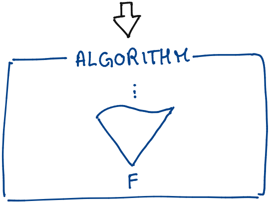


Decision procedures and proof theory



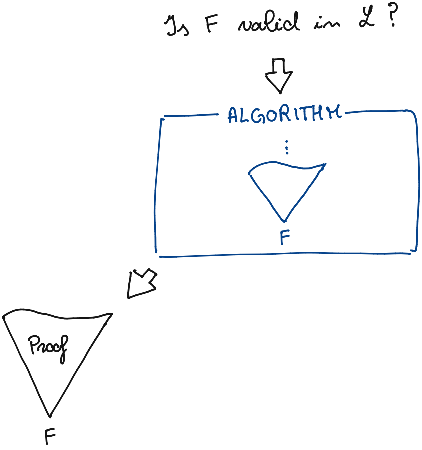
Decision procedures and proof theory

Is F valid in $\mathcal{L}$?



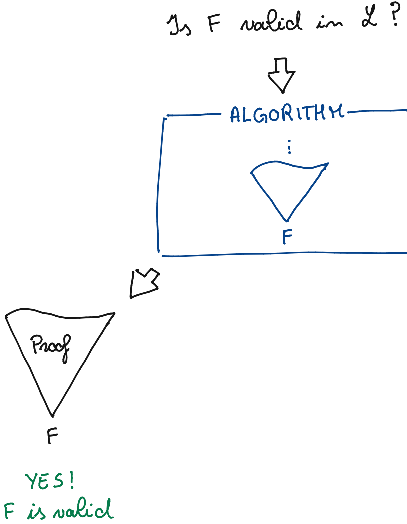
P proof system for $\mathcal{L}$

Decision procedures and proof theory



P proof system for $\mathcal{L}$

Decision procedures and proof theory



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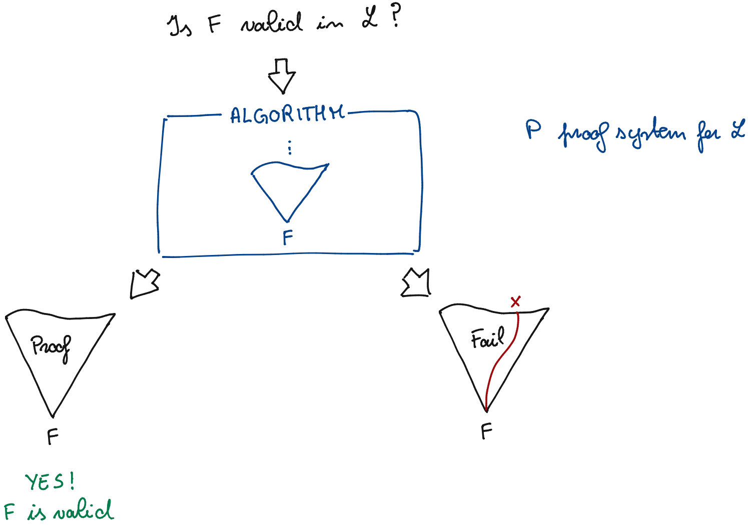
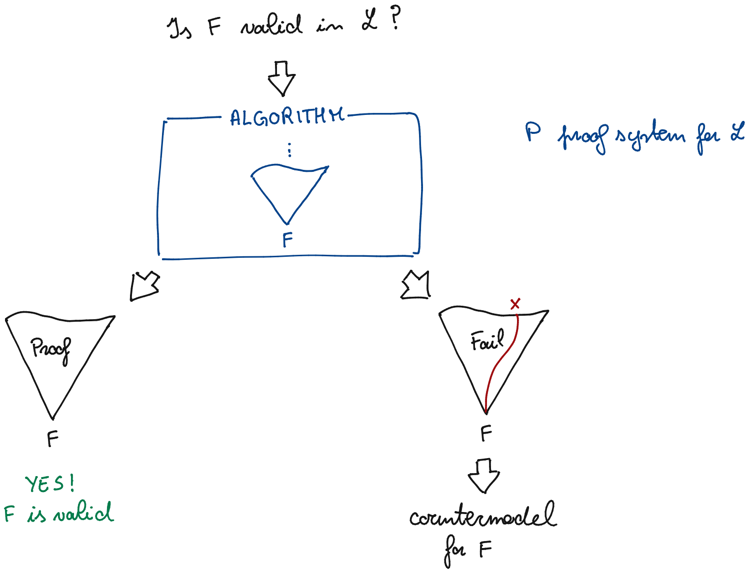


Decision procedures and proof theory



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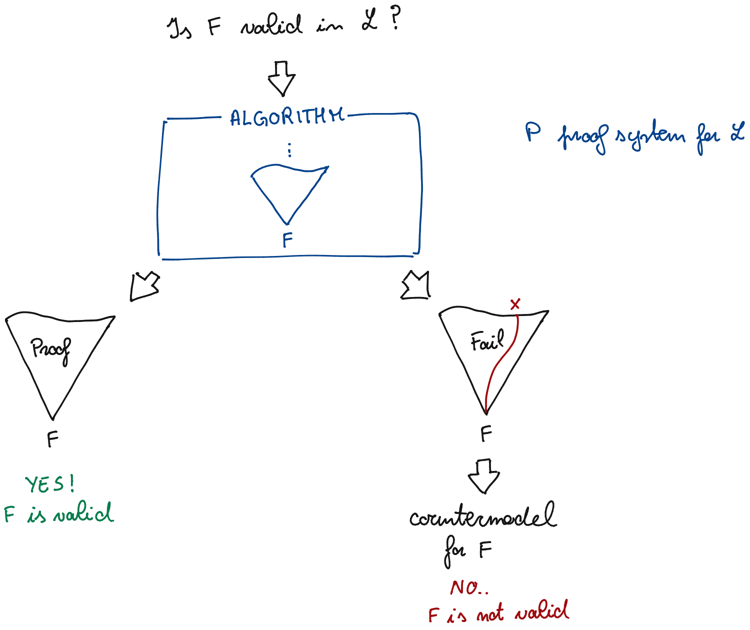


Desirable properties for proof search

- Termination of backward proof search

  ⇝ guaranteed by analyticity, and by the fact that the rules reduce the complexity of sequents

- Decision procedure by a single proof-search tree: it suffices to construct *one* derivation tree to check for derivability

  ⇝ guaranteed by invertibility of all the rules

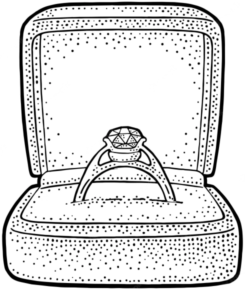
- Countermodel construction from (a leaf of) a failed branch

  ⇝ read the formulas in the antecedent as “true”, and those in the consequent as “false”

Summing up

	fml. interpr.	invertible rules	analyti-city	termination proof search	counterm. constr.	modu-larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a

G3-style sequent calculus for modal logic



The S5 cube of modal logics

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$$

$\mathcal{H}$K: axioms and rules from $\mathcal{H}$cp, plus:

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k $\Box(A \to B) \to (\Box A \to \Box B)$

$$\text{nec}\,\frac{A}{\Box A}$$

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d $\Box A \rightarrow \Diamond A$

t $\Box A \rightarrow A$

b $A \rightarrow \Box\Diamond A$

4 $\Box A \rightarrow \Box\Box A$

5 $\Diamond A \rightarrow \Box\Diamond A$

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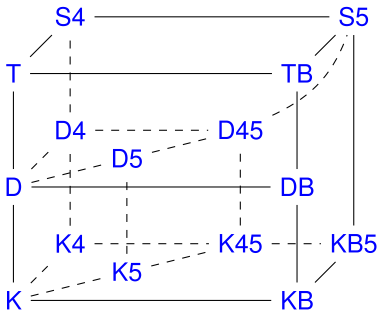

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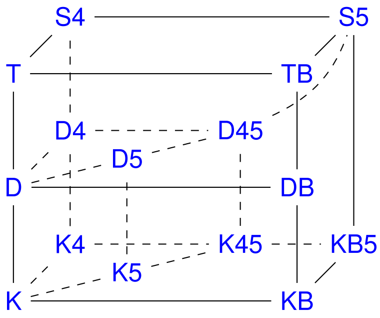

$\Gamma \vdash A \quad \rightsquigarrow \quad A$ is derivable from Γ in $\mathcal{H}$K

For $X \subseteq \{d, t, b, 4, 5\}$, $\Gamma \vdash_X A \quad \rightsquigarrow \quad A$ is derivable from Γ in $\mathcal{H}\mathrm{K} \cup X$

Kripke models

$$\mathcal{M} = \langle W, R, v \rangle$$

- W non-empty set of elements (*worlds*)
- R binary relation on W (*accessibility relation*)
- v *valuation function* $W \longrightarrow \mathcal{P}(Atm)$

Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box \Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \to xRz)$
5	$\Diamond A \to \Box \Diamond A$	Euclideaness	$\forall x \forall y \forall z ((xRy \wedge xRz) \to yRz)$

Notation. For $X \in \{d, t, b, 4, 5\}$ We denote by $\mathcal{X}$ the class of all models satisfying all conditions corresponding to the axioms of the logic X

Soundness and completeness

Satisfiability $\mathcal{M}, w \Vdash A$

$$\begin{array}{rcl} \mathcal{M}, w \Vdash p & \textit{iff} & p \in v(w) \\ & (..) & \\ \mathcal{M}, w \Vdash \Box A & \textit{iff} & \text{for all } u \text{ s.t. } wRu, u \Vdash A \\ \mathcal{M}, w \Vdash \Diamond A & \textit{iff} & \text{there exists } u \text{ s.t. } wRu \text{ and } u \Vdash A \end{array}$$

Validity in a model $\mathcal{M} \models A$ *iff* for all $w \in \mathcal{M}$, $\mathcal{M}, w \Vdash A$

Validity in a class of models $\models_{\mathcal{X}} A$ *iff* for all $\mathcal{M} \in \mathcal{X}$, $\mathcal{M} \models A$

Logical consequence

$\Gamma \models_{\mathcal{X}} A$ *iff* for all $\mathcal{M} \in \mathcal{X}$, for all $w \in \mathcal{M}$,
if $\mathcal{M}, w \Vdash B$ for all $B \in \Gamma$, *then* $\mathcal{M}, w \Vdash A$

Theorem. $\Gamma \vdash_{\mathsf{X}} A$ if and only if $\Gamma \models_{\mathcal{X}} A$ [Blackburn de Rijke, Venema, 2001]

Sequent calculi for (some) modal logics

For simplicity, we define the rules for the $\Box$-only fragment.

Some references: [Ohnishi, Matsumoto, 1957], [Fitting, 1983], [Takano, 1992]

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$$\mathbf{G3K} \;=\; \mathbf{G3cp} \;+\; \mathrm{k}\,\frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \text{ for } n \geq 0$$

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Notation. Given $\Sigma = B_1, \ldots, B_n$ (for $n \geq 0$), let $\Box\Sigma = \Box B_1, \ldots, \Box B_n$.
The rule k can be written as

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The rule k can be written as

$${\scriptstyle k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

$${\scriptstyle t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad {\scriptstyle 4}\,\frac{\Box\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad {\scriptstyle 45}\,\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

- Sequent calculus for T: **G3K** $\cup$ {t}
- Sequent calculus for S4: **G3cp** $\cup$ {4, t}
- Sequent calculus for S5: **G3cp** $\cup$ {45, t}

Sequent calculus for K

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Structural properties of **G3cp**

- Weakening and contraction are hp-admissible
- All propositional rules are hp-invertible (but not the rule k)
- Cut is admissible

Theorem (Soundness). If $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow \Delta$ then $\vdash i(\Gamma \Rightarrow \Delta)$.

Theorem (Completeness). If $\Gamma \vdash A$ then $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow A$.

Invertibility

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \;+\; \text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Rule k is not invertible

$$\mathbf{G3K} \quad = \quad \mathbf{G3cp} \; + \; \text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Rule k is not invertible

Consequently:

- One failed proof is not sufficient to ensure non-derivability
- Hence, in particular, it does not provide a countermodel
- Backward proof-search in **G3K** requires backtracking

Example

Is the sequent $\Box(p \vee q) \Rightarrow \Box p \vee \Box q$ derivable in **G3K**?

$$\dfrac{\Box(p \vee q) \Rightarrow \Box p, \Box q}{\Box(p \vee q) \Rightarrow \Box p \vee \Box q} \; \vee_R$$

Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no

End of content for today's lecture!

Questions?

1. Show that the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and the rule $\dfrac{\Rightarrow A}{\Rightarrow \Box A}$ are derivable in **G3K**.
2. We wish to show that **G3T** is not contraction-free complete [Goré, 1999]. Sequent calculus **G3T** adds the following two rules to **G3cp**:

$$\text{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta} \qquad \text{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

   We consider formula $\Box(p \wedge (\Box p \to \bot) \to \bot$, which is valid in T. Show that:
   a) The sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ is derivable in $\mathbf{G3T} \cup \{\mathrm{ctr_L}, \mathrm{ctr_R}\}$
   b) The sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ is not derivable in **G3T**
   c) If we substitute rule t with the following rule t′ in **G3T**, then sequent $\Rightarrow \Box(p \wedge (\Box p \to \bot) \to \bot$ becomes derivable (without contraction):

$$\text{t}'\,\frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

3. Next, we wish to show that **G3S5** is not cut-free complete. Sequent calculus **G3S5** adds the following rules to **G3cp**:

$$\text{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad 45\,\frac{\Box\Sigma \Rightarrow A, \Box\Pi}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi, \Delta}$$

   We consider formula $p \vee \Box(\Box p \to \bot)$, which is valid in S5. Show that:
   a) The sequent $p \vee \Box(\Box p \to \bot)$ is derivable in $\mathbf{G3S5} \cup \{\mathrm{cut}\}$
   b) The sequent $p \vee \Box(\Box p \to \bot)$ is not derivable in **G3S5**