Proof Theory of Modal Logic

Lecture 5
Beyond the S5-cube

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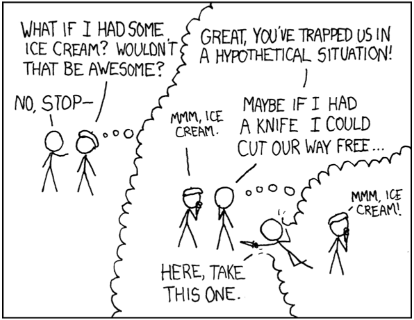
5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Today's lecture: Beyond the S5 – *cube*

- Conditional logics
- Labelled proof systems for conditional logics
- Structured proof systems for conditional logics

Conditionals

If A then B



Counterfactuals

- If I had some ice cream, then life would be awesome.
- If kangaroos had no tails, then they would topple over.
- If Sam saw a lunar eclipse, then she would no longer believe that Earth is flat.

Paradoxes of material implication

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A	B	$A \rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

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A	B	$A \to B$
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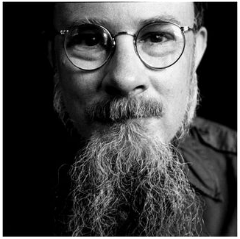
- Possible world models [Stalnaker, 1968], [Lewis, 1973]



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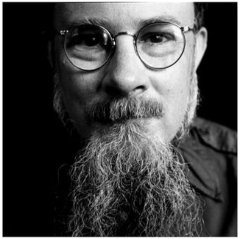
- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]
- Belief revision [Gärdenfors, 1978]
- Causal models [Galles and Pearl, 1998]
- Possible world models [Stalnaker, 1968], [Lewis, 1973], [Nute, 1980], [Chellas, 1975], [Burgess, 1981], and many more!



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Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

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$$\Diamond A := \neg \Box \neg A$$

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Conditional logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

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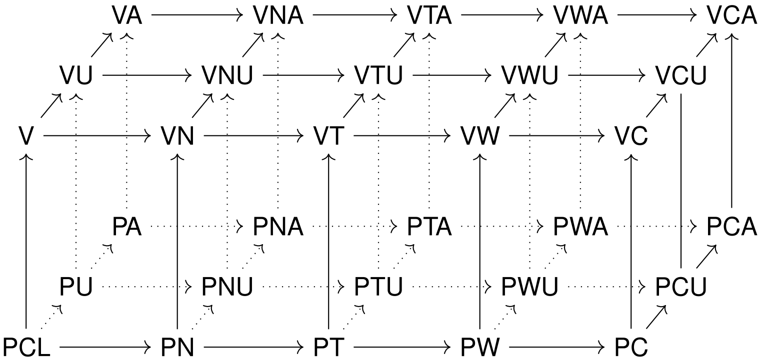
Conditional logics



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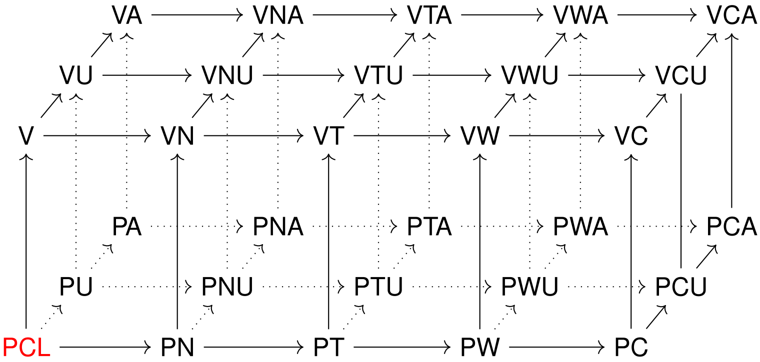

- Prototypical properties [KLM, 1990], [Burgess, 1981]

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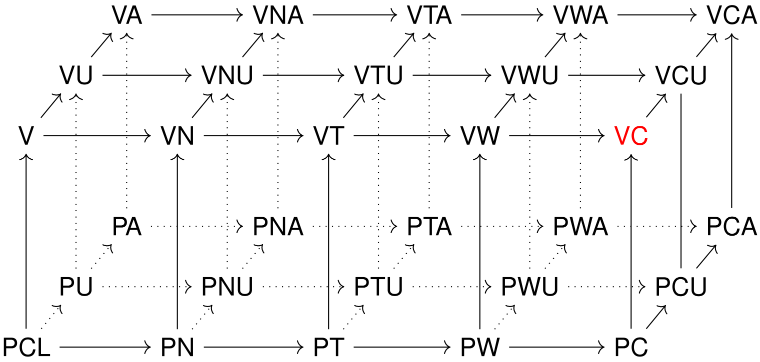

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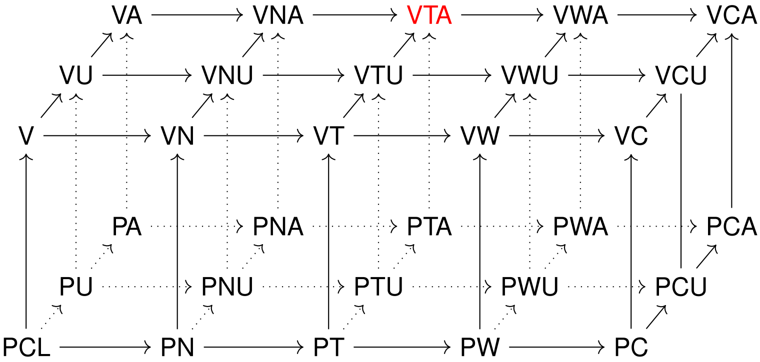

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- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets, 2006, 2008]

Axioms

PCL: classical propositional logic plus

rcea $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$ rck $\dfrac{A \rightarrow B}{(C > A) \rightarrow (C > B)}$

id $A > A$

r.and $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

cm $(A > B) \wedge (A > C) \rightarrow ((A \wedge C) > B)$

rt $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

or $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V: PCL plus

cv $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Extensions of PCL and V

n $\neg(\top > \bot)$

t $A \rightarrow \neg(A > \bot)$

w $(A > B) \rightarrow (A \rightarrow B)$

c $(A \wedge B) \rightarrow (A > B)$

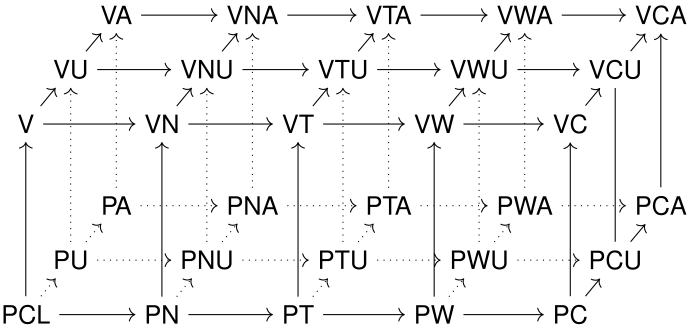
u_1 $(\neg A > \bot) \rightarrow (\neg(\neg A > \bot) > \bot)$

u_2 $\neg(A > \bot) \rightarrow ((A > \bot) > \bot)$

a_1 $(A > B) \rightarrow (C > (A > B))$

a_2 $\neg(A > B) \rightarrow (C > \neg(A > B))$

Possible-world semantics for conditionals



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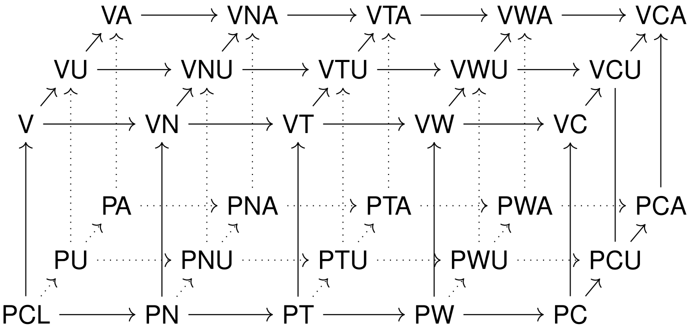

- Selection function semantics [Chellas, 1975]

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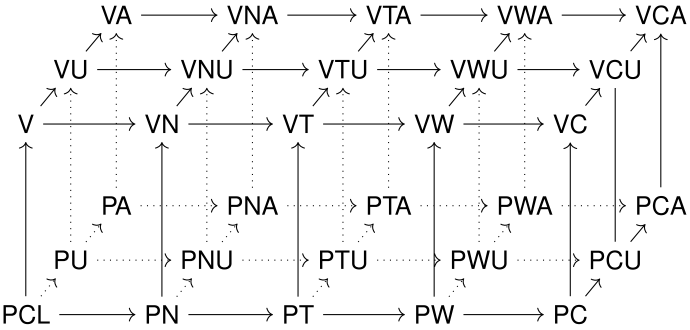

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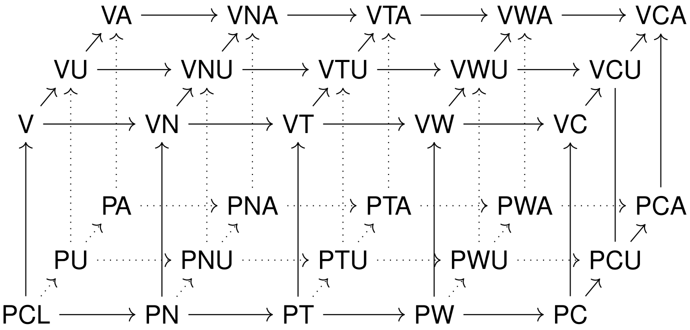

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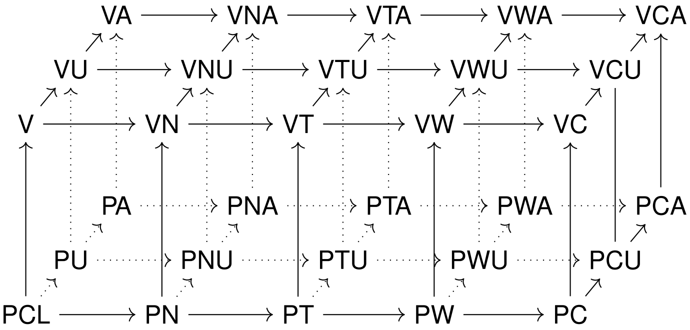

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- Neighbourhood semantics [Scott, 1970], [Montague, 1970]

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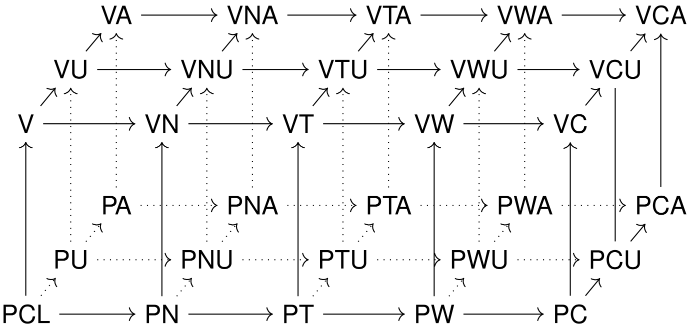

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Direct proof of soundness and completeness w.r.t. the axiomatization of PCL and extensions [G, Negri, Olivetti, 2021]

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$$

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$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \emptyset \notin N(x)$$

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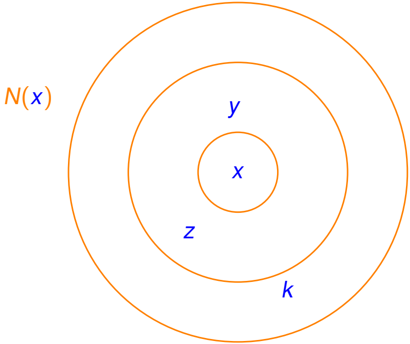

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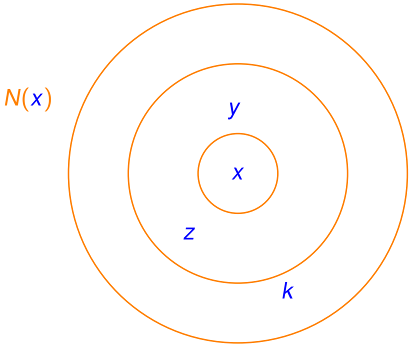

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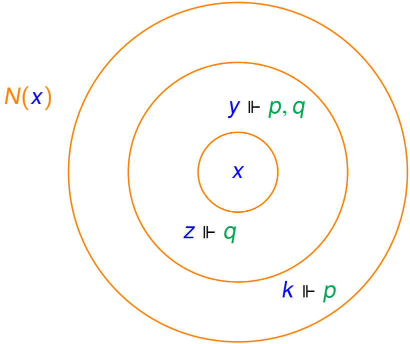

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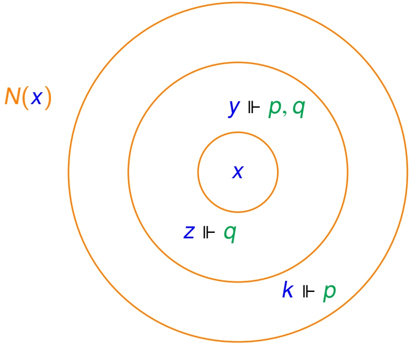
Nesting for all x , for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$



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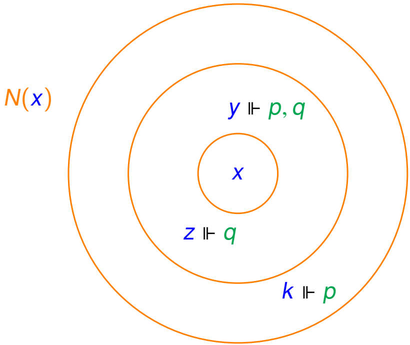

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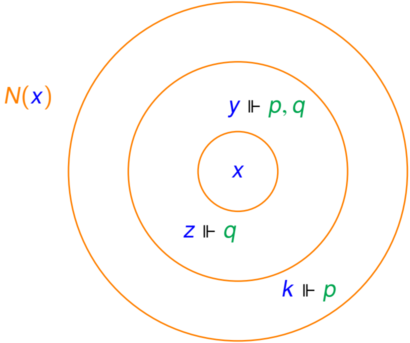

$x \Vdash p > q$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$, then there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

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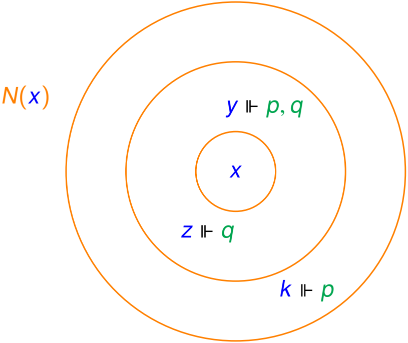

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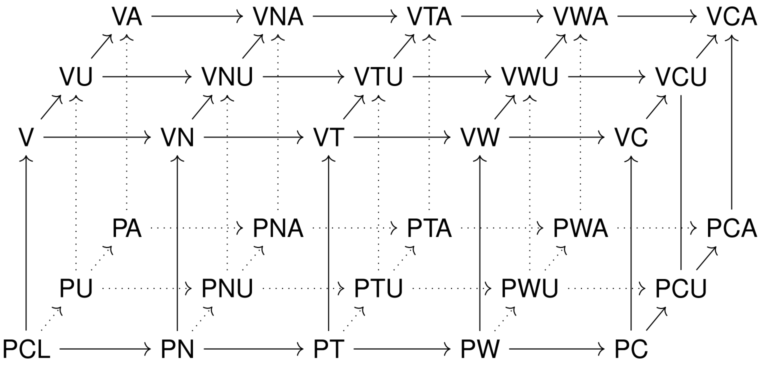
$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$

$\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Conditional logics



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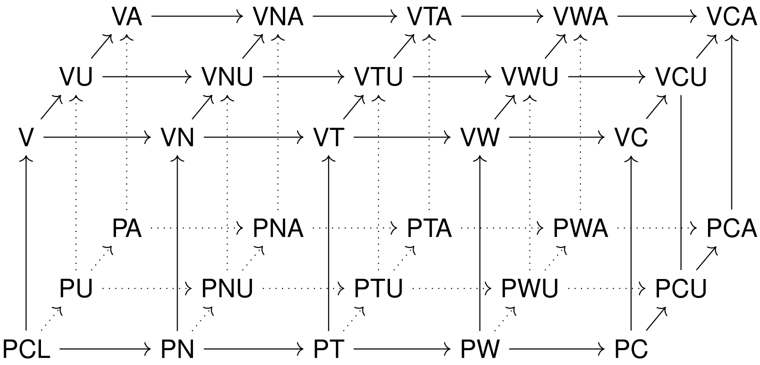
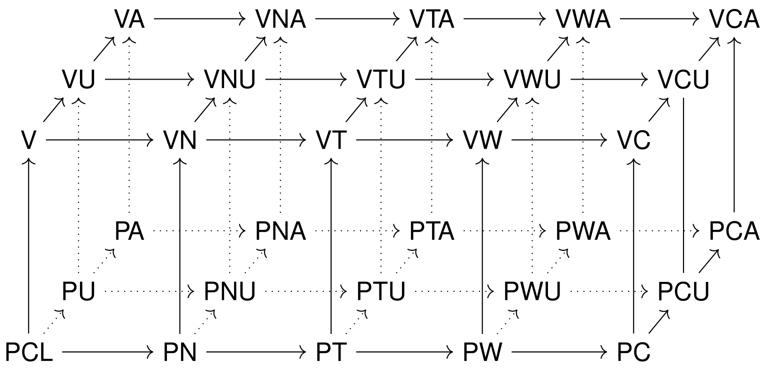


★ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V



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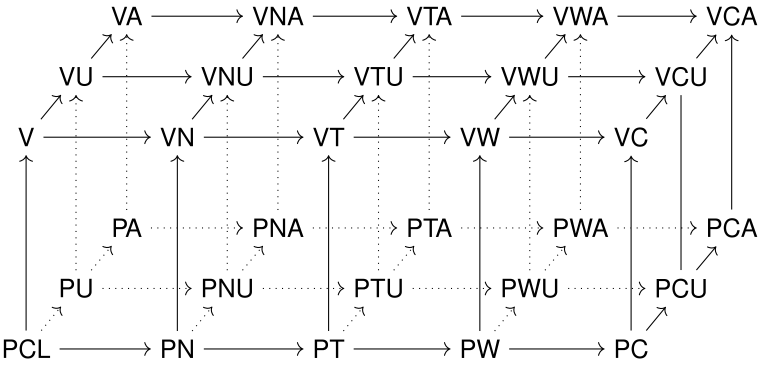

- ★ *Normality* For all x, $N(x) \neq \emptyset$. N
- ★ *Total reflexivity* For all x, there is $\alpha \in N(x)$ such that $x \in \alpha$. T
- ★ *Weak centering* For all x, $N(x) \neq \emptyset$ and for all $\alpha \in N(x)$, $x \in \alpha$. W
- ★ *Centering* For all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$. C
- ★ *Uniformity* For all x, y, $\bigcup N(y) = \bigcup N(x)$. U
- ★ *Absoluteness* For all x, y, $N(x) = N(y)$. A
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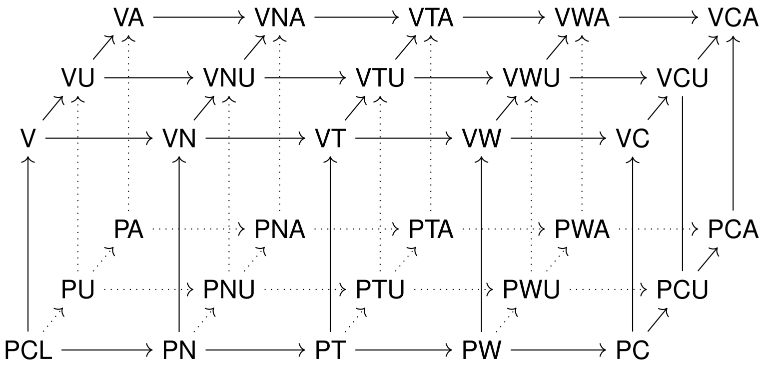
Labelled calculi for conditional logics



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- Rules for $\Box$

$$\Box_L \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\, y! \qquad \Box_R \frac{xRy, \mathcal{R}, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}$$

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- ☞ Rules for frame conditions, example: transitivity

$$\text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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- ☞ Labelled formulas
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Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \ldots$
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$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
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  - $x \Vdash_a A \mid B \rightsquigarrow$ "there is a $b \in N(x)$ such that $b \subseteq a$, $b \Vdash^{\exists} A$ and $b \Vdash^{\forall} A \rightarrow B$"

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Labelled rules

Labelled rules

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$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, xA > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \rightarrow B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

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- Rules for frame conditions: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

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Example

Axiom c $(p \wedge q) \rightarrow (p > q)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p, x : p}\,\text{init}
}{\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p}\,\Vdash^{\exists}_{R}
}{\ldots x : p \Rightarrow \{x\} \Vdash^{\exists} p}\,\text{Single}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{y \in \{x\}, \ldots, y : q, y : p \Rightarrow y : q}\,\text{init}
}{y \in \{x\}, \ldots, y : q \Rightarrow y : p \rightarrow q}\,\rightarrow_{R}
}{y \in \{x\}, \ldots, x : q \Rightarrow y : p \rightarrow q}\,\text{Repl}_1
}{\ldots x : q \Rightarrow \{x\} \Vdash^{\forall} p \rightarrow q}\,\Vdash^{\exists}_{R}
}{
\dfrac{
\dfrac{
\dfrac{
\{x\} \in a, \{x\} \subseteq a, a \in N(x), a \Vdash^{\exists} p, x : p, x : q \Rightarrow x \Vdash_a p \mid q
}{a \in N(x), a \Vdash^{\exists} p, x : p, x : q \Rightarrow x \Vdash_a p \mid q}\,\text{C}
}{x : p, x : q \Rightarrow x : p > q}\,>_{R}
}{
\dfrac{x : p \wedge q \Rightarrow x : p > q}{\Rightarrow x : (p \wedge q) \rightarrow (p > q)}\,\rightarrow_{R}
}\,\wedge_{L}
}\,\Vdash_{L}
$$

Main results [G, Negri and Olivetti, 2021]

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☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

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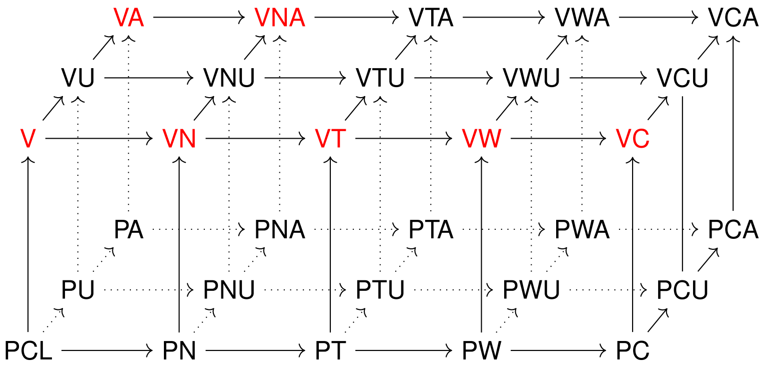
Structured proof systems for (some) Lewis' logics



Conditionals in a modal framework

To define our structured proof system, we change the language: instead of the conditional operator, we shall take as primitive the comparative plausibility operator, $\preceq$, also introduced by Lewis, which can be used to define the conditional.

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A \preceq B$$

"*A* is at least as plausible as *B*"

$$\Box A := \bot \preceq \neg A$$

$$A > B := (\bot \preceq A) \vee \neg((A \wedge \neg B) \preceq (A \wedge B))$$

The interdefinability of $>$ and $\preceq$ does not hold for all the systems, but only for Lewis' logics.

$$\mathcal{M}, x \models A \preceq B \text{ iff for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} B \text{ then } \alpha \Vdash^{\exists} A$$

Sequents with blocks

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Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

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$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k]$$

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Example

$$G_1, G_2 \Rightarrow D, [A, B \lhd C] \quad \rightsquigarrow \quad (G_1 \wedge G_2) \rightarrow \Big(D \vee ((A \preceq C) \vee (B \preceq C)) \Big)$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

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$$>_L\ \frac{\bot \preceq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \wedge \neg B) \preceq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

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$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preceq_L\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

$$>_L\ \frac{\bot \preceq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \wedge \neg B) \preceq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

$$A > B := (\bot \preceq A) \vee \neg((A \wedge \neg B) \preceq (A \wedge B))$$

☞ Rules for extensions: centering

$$\text{C}\ \frac{A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B}{A \preceq B, \Gamma \Rightarrow \Delta}$$

Examples

Axiom $(A \preceq B) \vee (B \preceq A)$

$$
\vee_R\dfrac{\preceq_R\dfrac{\preceq_R\dfrac{\text{com}\dfrac{\text{jump}\dfrac{\text{init}\dfrac{}{b \Rightarrow a, b}}{\Rightarrow a \preceq b, b \preceq a, [a, b \lhd b], [b \lhd a]} \qquad \text{jump}\dfrac{\text{init}\dfrac{}{a \Rightarrow a, b}}{\Rightarrow a \preceq b, b \preceq a, [a \lhd b], [a, b \lhd a]}}{\Rightarrow a \preceq b, b \preceq a, [a \lhd b], [b \lhd a]}}{\Rightarrow a \preceq b, b \preceq a, [a \lhd b]}}{\Rightarrow a \preceq b, b \preceq a}}{\Rightarrow (a \preceq b) \vee (b \preceq a)}
$$

Axiom c $(p \wedge q) \rightarrow (p > q)$

$$
\rightarrow_R\dfrac{\wedge_L\dfrac{>_R\dfrac{\text{c}\dfrac{\wedge_L\dfrac{\neg_L\dfrac{\text{init}\dfrac{}{p, p, q \Rightarrow [\bot \lhd p], q}}{p, \neg q, p, q \Rightarrow [\bot \lhd p]}}{p \wedge \neg q, p, q \Rightarrow [\bot \lhd p]} \qquad \text{init}\dfrac{}{p, q \Rightarrow [\bot \lhd p], p}}{(p \wedge \neg q) \preceq p, p, q \Rightarrow [\bot \lhd p]}}{p, q \Rightarrow p > q}}{p \wedge q \Rightarrow p > q}}{\Rightarrow (p \wedge q) \rightarrow (p > q)}
$$

Main results [G Lellmann, Olivetti, Pozzato, 2016]

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Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

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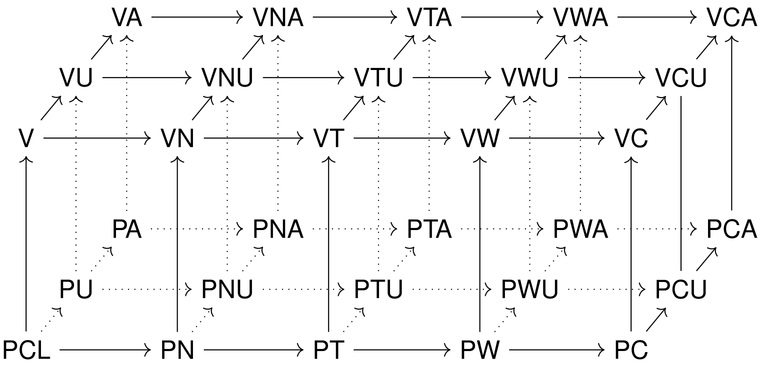
Proof. Show that if A is not provable, we can construct a finite countermodel for it (difficult). We need to show termination (easy).

Summing up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PC

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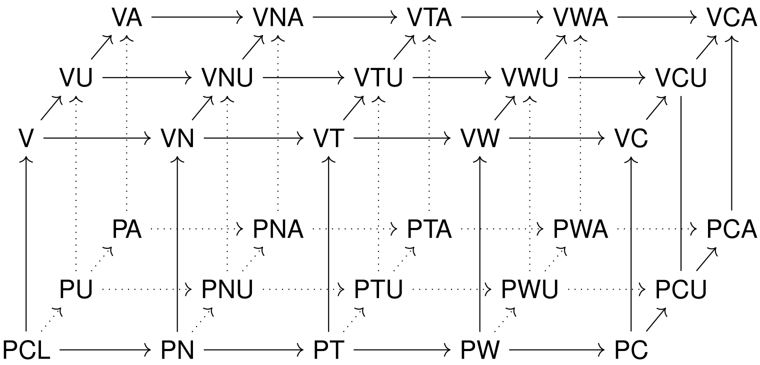

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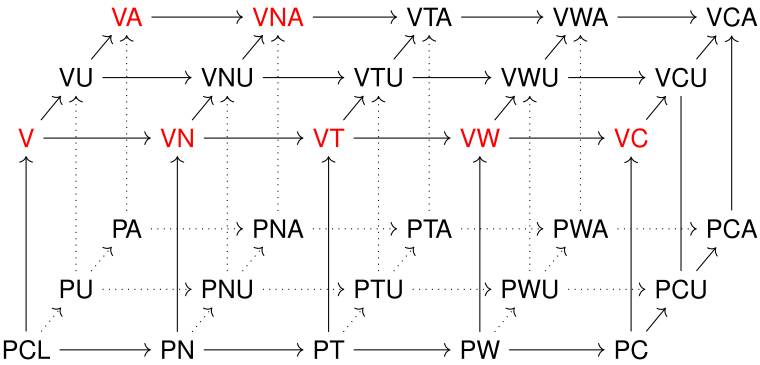

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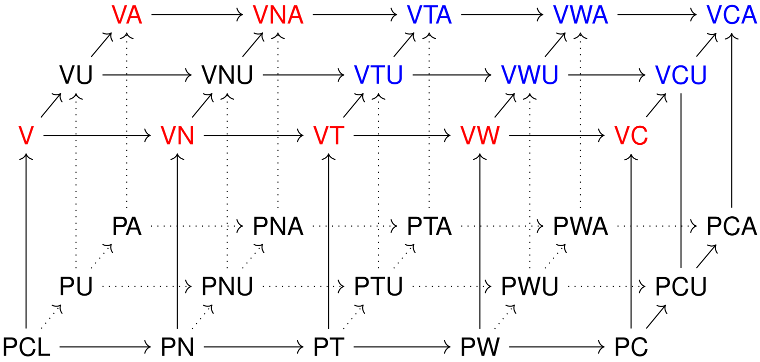

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- Hypersequent calculus with blocks for logics VTU, VWU, VCU, VTA, VWA, VCA [G, Lellmann, Olivetti, Pozzato, 2017]

Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$NK \cup X^{\diamond}$	yes	yes	yes	yes	yes, easy!	45-clause
$labK \cup X$	no	yes	yes	yes, for most	yes, easy!	yes
lab, cond	no	yes	easy	difficult	easy	yes
str, cond	yes	no	difficult	easy	difficult	no