Proof Theory of Modal Logic

Lecture 4
Semantic Completeness

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5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Recap

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$\mathsf{NK} \cup \mathsf{X}^{\diamond}$	yes	yes	yes	?	?	45-clause
$\mathsf{labK} \cup \mathsf{X}$	no	yes	yes	?	?	yes

Today's lecture: Semantic Completeness

- Semantic completeness for NK
- Semantic completeness for labK4

In the literature

For nested sequents: [Bruünler, 2009]: semantic completeness via terminating proof search for all the logics in the S5-cube

For labelled calculi:

- [Negri, 2005]: Minimality argument ensuring for some logics in the S5-cube (K, T, S4, S5);
- [Negri, 2014]: Semantic completeness via terminating proof search for intermediate logics;
- [Garg, Genovese and Negri, 2012]: Decision procedures via termination for multi-modal logics (without symmetry).

Semantic completeness for NK

NK: recap

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

NK: recap

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

$$\text{init}\,\frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\,\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

NK: recap

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For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

A nested sequent Γ is satisfied by an $\mathcal{M}$-map for Γ iff

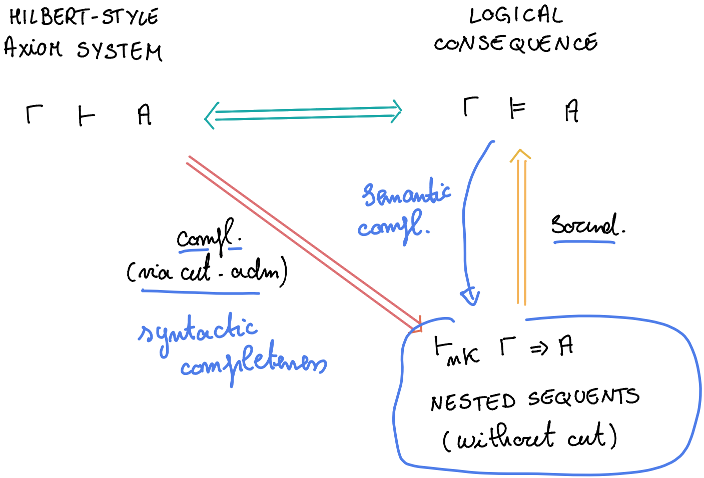
$$\mathcal{M}, f(\delta) \models B, \text{ for some } \delta \in tr(\Gamma), \text{ for some } B \in \delta$$

A nested sequent Γ is refuted by an $\mathcal{M}$-map for Γ iff

$$\mathcal{M}, f(\delta) \not\models B, \text{ for all } \delta \in tr(\Gamma), \text{ for all } B \in \delta$$

A nested sequent is valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$.

Roadmap


HILBERT-STYLE
AXIOM SYSTEM
Γ ⊢ A
LOGICAL
CONSEQUENCE
Γ ⊨ A
compl.
(via cut-adm)
syntactic
completeness
semantic
compl.
sound.
⊢MK Γ ⇒ A
NESTED SEQUENTS
(without cut)

Semantic completeness

Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

next slide.

Theorem (Semantic Completeness). If $\Gamma \models A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.

Proof. If $\bar{\Gamma} \vee A$ is not derivable in NK, then $\Gamma \not\models A$.

↓ there is $\mathcal{M}$ and $w \in W$ s.t. $\mathcal{M}, w \models \Gamma$ and $\mathcal{M}, w \not\models A$

Suppose $\bar{\Gamma} \vee A$ is not derivable.
By PorC Lemma, there is model $\mathcal{M}^x$ and $\mathcal{M}^x$-map f^x s.t.

- $\mathcal{M}^x, f^x(\delta) \not\models \bar{G}$ for all $\bar{G} \in \bar{\Gamma}$ and
- $\mathcal{M}^x, f^x(\delta) \not\models A$
- $\mathcal{M}^x, f^x(\delta) \models G$ for all $G \in \Gamma$

$\Gamma \not\models A$

Proof or countermodel

Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

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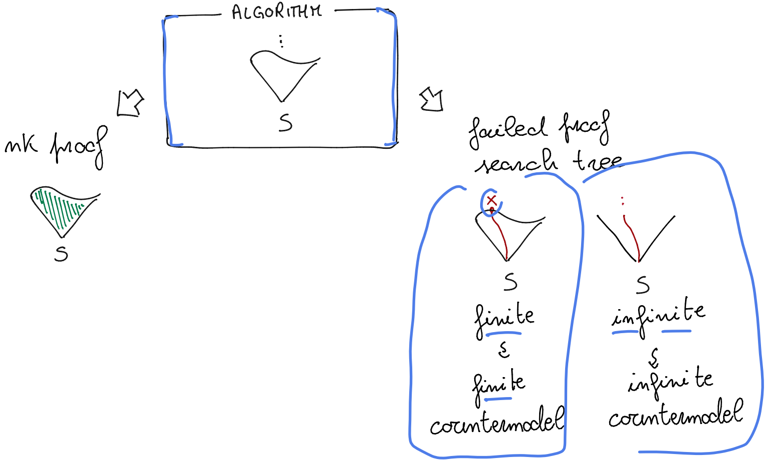
Proof (sketch). Algorithm implementing proof search in nK



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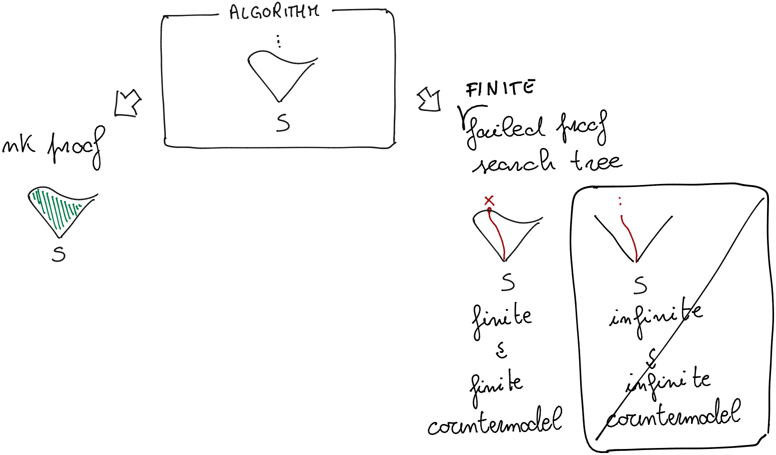

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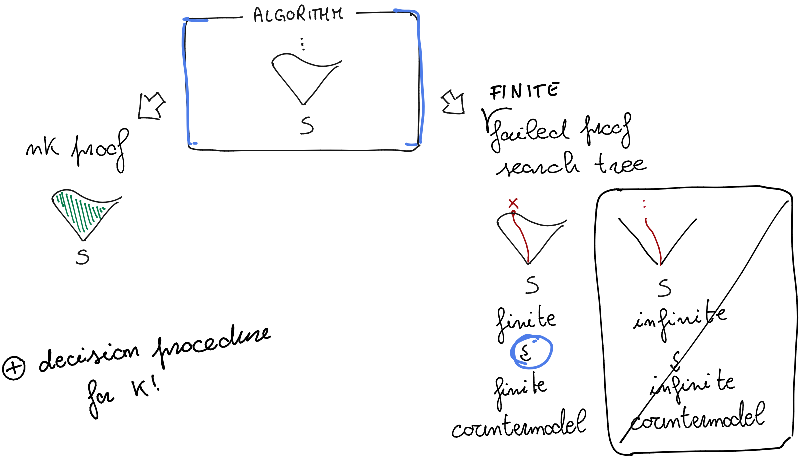

Termination

$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$\begin{array}{c} \vdots \\ \hline \Diamond p, [q, p, p] \\ \hline \Diamond p, [q, p] \\ \hline \Diamond p, [q] \end{array} \begin{array}{l} \\ \Diamond \\ \Diamond \end{array} \qquad \begin{array}{c} \vdots \\ \Diamond (p \vee r), [p \vee r, p, r, q] \\ \hline \Diamond (p \vee r), [p, r, q] \\ \hline \Diamond (p \vee r), [p \vee r, q] \\ \hline \Diamond (p \vee r), [q] \end{array} \begin{array}{l} \\ \Diamond \\ \vee \\ \Diamond \end{array}$$

A cumulative version of NK

Rules of NK^c

$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A \wedge B, A\} \quad \Gamma\{A \wedge B, B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A \vee B, A, B\}}{\Gamma\{A \vee B\}}$$

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Proposition. NK and NK^c are equivalent.

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Proposition. NK and NK^c are equivalent.

The set-nested sequent of a nested sequent $A_1, \ldots, A_n, [\Delta_1], \ldots, [\Delta_m]$ is the underlying set $A_1, \ldots, A_n, [\Lambda_1], \ldots, [\Lambda_m]$, where $\Lambda_1, \ldots, \Lambda_m$ are the set-nested sequents of $\Delta_1, \ldots, \Delta_m$.

A cumulative version of NK

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A rule application is redundant if the set-nested sequent of one of its premisses is the same as the set-nested sequent of its conclusion.

Avoid redundant applications of the rules!

$$\Gamma = \dfrac{\Delta = \dfrac{\dfrac{\Diamond(p \vee r), [p \vee r, p \vee r, p, r, q]}{\Diamond(p \vee r), [p \vee r, p, r, q]}\,\Diamond \leftarrow \text{redundant!!}}{\Diamond(p \vee r), [p \vee r, q]}\,\vee}{\Diamond(p \vee r), [q]}\,\Diamond$$

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set-NS of Γ = set-NS of Δ

$$\frac{\Gamma = \Box p, [p], [p]}{\Delta = \Box p, [p]}\ \Box \leftarrow \text{redundant!!}$$

$$\frac{\Box p, [p]}{\Box p}\ \Box$$

A terminating proof search algorithm for NK^c

Is Γ derivable in NK^c?

0. Place $\Gamma_0 = \Gamma$ at the root of $\mathcal{T}$.
1. For every topmost nested sequent Γ_i of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: $\wedge, \vee, \diamond$.
2. If every topmost nested sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ Γ_0 is derivable in NK^c.
3. Otherwise, pick a non-initial topmost nested sequent Γ_k of $\mathcal{T}$.
   a) If there is a non-redundant $\Box$-rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ Γ_0 is not derivable in NK^c.

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constructed using the algorithm

Theorem (Termination). Root-first proof search in NK^c terminates in a finite number of steps.

Constructing a countermodel

Lemma. If proof search terminates in step 3, then there is an $\mathcal{M}$-map for Γ_0 such that Γ_0 is refuted by the $\mathcal{M}$-map.

Proof. Consider Γ_k, the non-initial topmost nested sequent where the algorithm stopped. We define the model $\mathcal{M}^\times = \langle W^\times, R^\times, v^\times \rangle$ as follows:

- $W^\times = \{\delta \mid \delta \in tr(\Gamma_k)\}$
- $\delta R^\times \gamma$ iff γ is a child of δ in $tr(\Gamma_k)$
- $v^\times(\delta) = \{p \mid \bar{p} \in \delta\}$

Moreover, let $f^\times$ be the $\mathcal{M}^\times$-map for Γ_k defined by setting $f^\times(\delta) = \delta$, for every $\delta \in tr(\Gamma_k)$.

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We have to prove that $\mathcal{M}^\times$ is a Kripke model (easy) and that $f^\times$ is an $\mathcal{M}^\times$-map (also easy).

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Next, we need to prove that, for all formulas A:

$$\text{if } A \in \delta \in tr(\Gamma_k) \text{ then } \mathcal{M}^\times, f^\times(\delta) \not\models A$$

$\Box B \in \delta$

$\mathcal{M}^\times, f^\times(\delta) \not\models \Box B$

Example NK^c

$$\Gamma_K = \Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar p, q] \quad (\text{worlds } 1, 2, 3)$$

$$\text{init}\ \dfrac{}{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar q, q]}$$

$$\wedge\ \dfrac{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar p, q] \qquad \Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar q, q]}{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar p \wedge \bar q, q]}$$

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$$\Box\ \dfrac{\Diamond(\bar p \wedge \bar q), [\bar q, p], [q]}{\Diamond(\bar p \wedge \bar q), [\bar q, p], \Box q}$$

$$\text{init}\ \dfrac{}{\Diamond(\bar p \wedge \bar q), [\bar p, p], \Box q}$$

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$$\Box\ \dfrac{\Diamond(\bar p \wedge \bar q), [p], \Box q}{\Diamond(\bar p \wedge \bar q), \Box p, \Box q}$$

$$\vee\ \dfrac{\Diamond(\bar p \wedge \bar q), \Box p, \Box q}{\Diamond(\bar p \wedge \bar q), \Box p \vee \Box q}$$

$$\vee\ \dfrac{\Diamond(\bar p \wedge \bar q), \Box p \vee \Box q}{\Diamond(\bar p \wedge \bar q) \vee (\Box p \vee \Box q)}$$

$F = \Diamond(\bar p \wedge \bar q) \vee (\Box p \vee \Box q)$

$W^x = \{1, 2, 3\}$

$R^x = \{(1,2), (1,3)\}$

$v^x(1) = \emptyset$

$v^x(2) = q \qquad v^x(3) = p$

$2 \Vdash q$, $2 \nVdash p$

$3 \Vdash p$, $3 \nVdash q$

$1 \nVdash \Diamond(\bar p \wedge \bar q)$

$1 \nVdash \Box p \vee \Box q$

$1 \nVdash F$

Semantic completeness for labK4

Rules of labK4, a proof system for K4

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_L\ \frac{\mathcal{R}, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad rlr\wedge\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_L\ \frac{\mathcal{R}, x{:}A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B}$$

$$\rightarrow_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{x{:}A, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B}$$

$$\Box_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

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$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

y fresh means $y \neq x$ and y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Sources of non-termination

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q \Rightarrow}\,\Box_L$$

labK4^c, a cumulative version of labK4

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_L\ \frac{\mathcal{R}, x{:}A \wedge B, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B} \qquad \vee_L\ \frac{\mathcal{R}, x{:}A, x{:}A \vee B, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}B, x{:}A \vee B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_R\ \frac{x{:}A, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B} \qquad \rightarrow_L\ \frac{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}B, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}$$

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$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

y fresh means $y \neq x$ and y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Redundant rule applications

Intuitively: A rule application R is redundant at a sequent $\mathcal{S}$ if $\mathcal{S}$ already contains the formulas that would be introduced in one premiss of R.

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Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

- (tr) If xRy and yRz occur in $\mathcal{R}$, then xRz occurs in $\mathcal{R}$;
- ($\wedge_L$) If $x{:}A \wedge B$ occurs in Γ, then both $x{:}A$ and $x{:}B$ occur in Γ;
- ($\wedge_R$) If $x{:}A \wedge B$ occurs in Δ, then $x{:}A$ occurs in Δ or $x{:}B$ occur in Δ;
- (..)
- ($\Box_L$) If xRy occurs in $\mathcal{R}$ and $x{:}\Box A$ occurs in Γ, then $y{:}A$ occurs in Γ;
- ($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

Avoid redundant applications of the rules!

Sources of non-termination

$$
\begin{array}{rc}
 & \vdots \\
\Box_R & \overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p} \\
\Diamond_R & \overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p} \\
\text{tr} & \overline{2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p} \\
\Box_R & \overline{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p} \\
\Diamond_R & \overline{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p} \\
\text{tr} & \overline{1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p} \\
\Box_R & \overline{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p} \\
\Diamond_R & \overline{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot} \\
\Box_R & \overline{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot} \\
\vee_R & \overline{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}
\end{array}
$$

Limit applications of $\Box_R$ and $\Diamond_L$

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A} \; y \text{ fresh} \qquad \Diamond_L \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta} \; y \text{ fresh}$$

Limit applications of $\Box_R$ and $\Diamond_L$

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh} \qquad \Diamond_L \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

Limit applications of $\Box_R$ and $\Diamond_L$

$k \sim x$ iff $\{G \mid k{:}G \in \Gamma\} = \{E \mid x{:}E \in \Gamma\}$ and $\{D \mid k{:}D \in \Delta\} = \{F \mid x{:}F \in \Delta\}$

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh} \qquad \Diamond_L \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

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replaced by

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then either

a) there is a k such that kRx occurs in $\mathcal{R}$ and $k \sim x$; otherwise

b) there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

If *a*) holds, we say that x is a copy of k at $\mathcal{S}$

A terminating proof search algorithm for labK4

Is $x{:}\Gamma \Rightarrow x{:}A$ derivable in labK4?

0. Place $\mathcal{S}_0 = x{:}\Gamma \Rightarrow x{:}A$ at the root of $\mathcal{T}$.
1. For every topmost labelled sequent $\mathcal{S}_i$ of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: tr, $\wedge_L, \wedge_R, \vee_L, \vee_R, \rightarrow_L, \rightarrow_R, \Box_L, \Diamond_R$.
2. If every topmost labelled sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is derivable in labK4.
3. Otherwise, pick a non-initial topmost labelled sequent $\mathcal{S}_k$ of $\mathcal{T}$.
   a) If there are non-redundant $\Box_R$- or $\Diamond_L$- rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is not derivable in labK4.

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2. If every topmost labelled sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is derivable in labK4.
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Theorem (Termination). Root-first proof search in labK4^c terminates in a finite number of steps.

Validity of sequents

Given a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$, and a model $\mathcal{M} = \langle W, R, v \rangle$, let $\text{Lb}(\mathcal{S}) = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \text{Lb}(\mathcal{S}) \rightarrow W$ (interpretation).

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Satisfiability of labelled formulas at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash xRy \quad \textit{iff} \quad \rho(x) R \rho(y)$$

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Satisfiability of sequents at $\mathcal{M}$ under ρ (φ is xRy or $x{:}A$):

$$\mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff}$$

if for all $\varphi \in \mathcal{R} \cup \Gamma$ it holds that $\mathcal{M}, \rho \Vdash \varphi$,

then for some $x{:}D \in \Delta$ it holds that $\mathcal{M}, \rho \Vdash x{:}D$.

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then for some $x{:}D \in \Delta$ it holds that $\mathcal{M}, \rho \Vdash x{:}D$.

A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ has a countermodel iff there are $\mathcal{M}$, ρ such that:

- $\mathcal{M}, \rho \models \varphi$, for all $\varphi \in \mathcal{R} \cup \Gamma$, and
- $\mathcal{M}, \rho \not\models x{:}D$, for all $x{:}D \in \Delta$.

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Validity of sequents in a class of frames $\mathcal{X}$:

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X},\ \mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta$$

Constructing a countermodel

Lemma. If proof search terminates in step 3, then $\mathcal{S}_0$ has a countermodel.

Proof. Consider $\mathcal{S}_k$, the non-initial topmost nested sequent where the algorithm stopped. We define the model $\mathcal{M}^\times = \langle W^\times, R^\times, v^\times \rangle$ as follows:

- $W^\times = \{x \mid x \text{ occurs in } \mathcal{S}\}$;
- To define $R^\times$, first define:
  - $xR_1^\times y$ iff xRy occurs in $\mathcal{R}$;
  - $xR_2^\times k$ iff x is a $\Box$-copy (or $\Diamond$-copy) of k.

  $\mathcal{R}^\times$ is the transitive closure of $R_1^\times \cup R_2^\times$.
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Take $\rho^\times(x) = x$, for each label x occurring in $\mathcal{S}$. Then:

- If $x{:}A$ occurs in Γ, then $\mathcal{M}^\times, \rho^\times \models x{:}A$
- If $x{:}A$ occurs in Δ, then $\mathcal{M}^\times, \rho^\times \not\models x{:}A$

Example

fail

$$\overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}$$

$$\Diamond_R \; \overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}$$

$$\text{tr} \; \overline{2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}$$

$$\Box_R \; \overline{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p}$$

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$$\text{tr} \; \overline{1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p}$$

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$$\Diamond_R \; \overline{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}$$

$$\Box_R \; \overline{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}$$

$$\vee_R \; \overline{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}$$

Example

$$\frac{}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}\ \Box_R$$

$$\frac{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}\ \Diamond_R$$

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$$\frac{2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p}\ \Box_R$$

$$\frac{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p}{0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p}\ \Diamond_R$$

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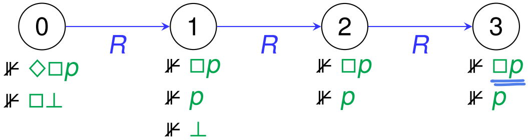
$$\frac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p}{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}\ \Diamond_R$$

$$\frac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}\ \Box_R$$

$$\frac{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}\ \vee_R$$



Example

$$\Box_R \; \overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}$$

$$\Diamond_R \; \overline{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}$$

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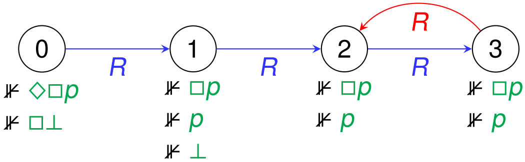
$$\vee_R \; \overline{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}$$



Summing up

semantic completeness

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$NK \cup X^{\diamond}$	yes	yes	yes	yes	yes, easy!	45-clause
$labK \cup X$	no	yes	yes	yes, for most	yes, easy!	yes

End of content for today's lecture!

Questions?

Exercises

1. Check whether $\Diamond\Box(p \vee \Box(p \rightarrow \bot))$ is valid in K4 using the terminating algorithm for labK4. If the formula is not valid, produce a countermodel.

2. Let $\mathcal{M}^\times$ be the countermodel for a labelled sequent $\mathcal{S}$. Verify that $\mathcal{M}^\times$ satisfies the frame condition of transitivity.
   Then, for $\rho^\times(x) = x$, for each label x occurring in $\mathcal{S}$, verify that the Truth Lemma holds, for the cases:
   - If $x{:}\Box A$ occurs in Γ, then $\mathcal{M}^\times, \rho^\times \models x{:}\Box A$
   - If $x{:}\Box A$ occurs in Δ, then $\mathcal{M}^\times, \rho^\times \not\models x{:}\Box A$