Proof Theory of Modal Logic

Lecture 4
Semantic Completeness

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5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Recap

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
NK ∪ $X^{\Diamond}$	yes	yes	yes	?	?	45-clause
labK ∪ X	no	yes	yes	?	?	yes

Today's lecture: Semantic Completeness

- Semantic completeness for NK
- Semantic completeness for labK4

In the literature

For nested sequents: [Bruünler, 2009]: semantic completeness via terminating proof search for all the logics in the S5-cube

For labelled calculi:

- [Negri, 2005]: Minimality argument ensuring for some logics in the S5-cube (K, T, S4, S5);
- [Negri, 2014]: Semantic completeness via terminating proof search for intermediate logics;
- [Garg, Genovese and Negri, 2012]: Decision procedures via termination for multi-modal logics (without symmetry).

Semantic completeness for NK

NK: recap

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

NK: recap

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$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

NK: recap

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For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

A nested sequent Γ is satisfied by an $\mathcal{M}$-map for Γ iff

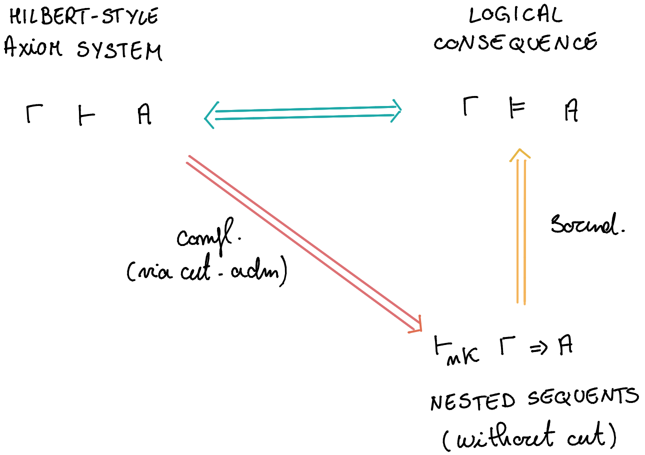
$$\mathcal{M}, f(\delta) \models B, \ \text{for some } \delta \in tr(\Gamma), \ \text{for some } B \in \delta$$

A nested sequent Γ is refuted by an $\mathcal{M}$-map for Γ iff

$$\mathcal{M}, f(\delta) \not\models B, \ \text{for all } \delta \in tr(\Gamma), \ \text{for all } B \in \delta$$

A nested sequent is valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$.

Roadmap


HILBERT-STYLE AXIOM SYSTEM
$\Gamma \vdash A$
LOGICAL CONSEQUENCE
$\Gamma \vDash A$
compl.
(via cut-adm)
Sound.
$\vdash_{MK} \Gamma \Rightarrow A$
NESTED SEQUENTS
(without cut)

Semantic completeness

Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

Theorem (Semantic Completeness). If $\Gamma \models A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.

Proof or countermodel

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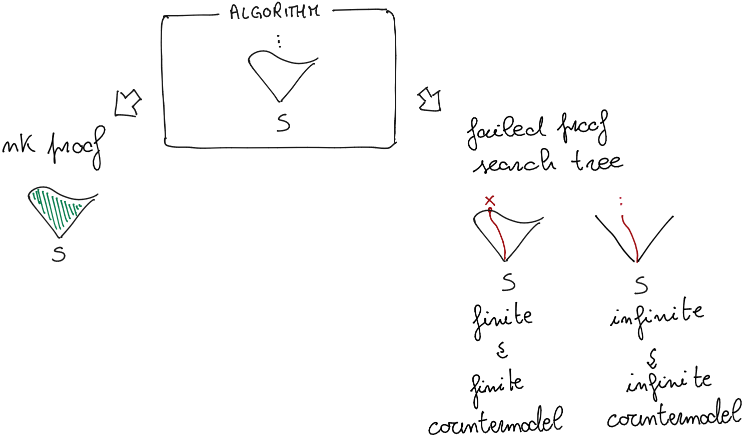
Proof (sketch). Algorithm implementing proof search in nK



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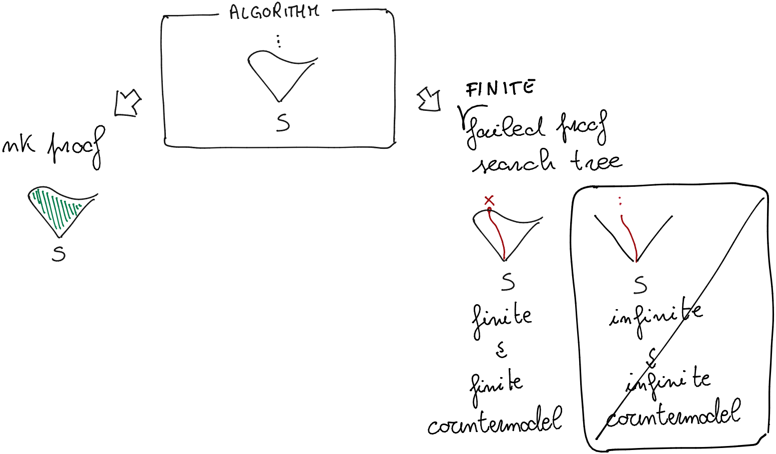

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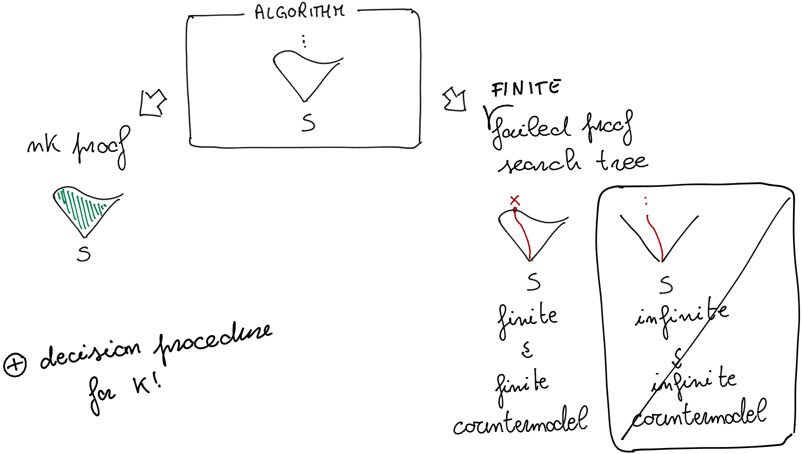

Termination

$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

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A cumulative version of NK

Rules of NK^c

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Proposition. NK and NK^c are equivalent.

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Proposition. NK and NK^c are equivalent.

The set-nested sequent of a nested sequent $A_1, \ldots, A_n, [\Delta_1], \ldots, [\Delta_m]$ is the underlying set $A_1, \ldots, A_n, [\Lambda_1], \ldots, [\Lambda_m]$, where $\Lambda_1, \ldots, \Lambda_m$ are the set-nested sequents of $\Delta_1, \ldots, \Delta_m$.

A cumulative version of NK

Rules of $\mathrm{NK^c}$

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A rule application is redundant if the set-nested sequent of one of its premisses is the same as the set-nested sequent of its conclusion.

Avoid redundant applications of the rules!

A terminating proof search algorithm for NK^c

Is Γ derivable in NK^c?

0. Place $\Gamma_0 = \Gamma$ at the root of $\mathcal{T}$.
1. For every topmost nested sequent Γ_i of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: $\wedge, \vee, \Diamond$.
2. If every topmost nested sequent of $\mathcal{T}$ is initial, terminate.
   $\rightsquigarrow$ Γ_0 is derivable in NK^c.
3. Otherwise, pick a non-initial topmost nested sequent Γ_k of $\mathcal{T}$.
   a) If there is a non-redundant $\Box$-rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ Γ_0 is not derivable in NK^c.

A terminating proof search algorithm for NK^{c}

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Theorem (Termination). Root-first proof search in NK^{c} terminates in a finite number of steps.

Lemma. If proof search terminates in step 3, then there is an $\mathcal{M}$-map for Γ_0 such that Γ_0 is refuted by the $\mathcal{M}$-map.

Proof. Consider Γ_k, the non-initial topmost nested sequent where the algorithm stopped. We define the model $\mathcal{M}^\times = \langle W^\times, R^\times, v^\times \rangle$ as follows:

- $W^\times = \{\delta \mid \delta \in tr(\Gamma_k)\}$
- $\delta R^\times \gamma$ iff γ is a child of δ in $tr(\Gamma_k)$
- $v^\times(\delta) = \{p \mid \bar{p} \in \delta\}$

Moreover, let $f^\times$ be the $\mathcal{M}^\times$-map for Γ_k defined by setting $f^\times(\delta) = \delta$, for every $\delta \in tr(\Gamma_k)$.

Constructing a countermodel

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We have to prove that $\mathcal{M}^\times$ is a Kripke model (easy) and that $f^\times$ is an $\mathcal{M}^\times$-map (also easy).

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Next, we need to prove that, for all formulas A:

$$\text{if } A \in \delta \in tr(\Gamma_k) \text{ then } \mathcal{M}^\times, f^\times(\delta) \not\models A$$

Example

$$
\vee\;\dfrac{\vee\;\dfrac{\Box\;\dfrac{\Diamond\;\dfrac{\wedge\;\dfrac{\text{init}\;\dfrac{}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{p},p],\Box q}\qquad \Box\;\dfrac{\Diamond\;\dfrac{\wedge\;\dfrac{\Diamond(\bar{p}\wedge\bar{q}),[\bar{q},p],[\bar{p},q]\qquad \text{init}\;\dfrac{}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{q},p],[\bar{q},q]}}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{q},p],[\bar{p}\wedge\bar{q},q]}}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{q},p],[q]}}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{q},p],\Box q}}{\Diamond(\bar{p}\wedge\bar{q}),[\bar{p}\wedge\bar{q},p],\Box q}}{\Diamond(\bar{p}\wedge\bar{q}),[p],\Box q}}{\Diamond(\bar{p}\wedge\bar{q}),\Box p,\Box q}}{\Diamond(\bar{p}\wedge\bar{q}),\Box p\vee\Box q}}{\Diamond(\bar{p}\wedge\bar{q})\vee(\Box p\vee\Box q)}
$$

Semantic completeness for labK4

Rules of labK4, a proof system for K4

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_L\ \frac{\mathcal{R}, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad rlr\wedge\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

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$$\rightarrow_L\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{x{:}A, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B}$$

$$\Box_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

$$\Diamond_L\ \frac{xRy, \mathcal{R}, y{:}A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \Diamond_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A, y{:}A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

y fresh means $y \neq x$ and y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Sources of non-termination

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{1{:}2, 1{:}\Box q, 2{:}q, 2{:}q, 2{:}q \Rightarrow}\ \Box_L}{1{:}2, 1{:}\Box q, 2{:}q, 2{:}q \Rightarrow}\ \Box_L}{1{:}2, 1{:}\Box q, 2{:}q \Rightarrow}\ \Box_L}{1{:}2, 1{:}\Box q \Rightarrow}\ \Box_L$$

labK4^c, a cumulative version of labK4

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_L\ \frac{\mathcal{R}, x{:}A \wedge B, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_R\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B} \qquad \vee_L\ \frac{\mathcal{R}, x{:}A, x{:}A \vee B, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}B, x{:}A \vee B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_R\ \frac{x{:}A, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B} \qquad \rightarrow_L\ \frac{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}B, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$\Box_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

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Redundant rule applications

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Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

- (tr) If xRy and yRz occur in $\mathcal{R}$, then xRz occurs in $\mathcal{R}$;
- ($\wedge_L$) If $x{:}A \wedge B$ occurs in Γ, then both $x{:}A$ and $x{:}B$ occur in Γ;
- ($\wedge_R$) If $x{:}A \wedge B$ occurs in Δ, then $x{:}A$ occurs in Δ or $x{:}B$ occur in Δ;
- (..)
- ($\Box_L$) If xRy occurs in $\mathcal{R}$ and $x{:}\Box A$ occurs in Γ, then $y{:}A$ occurs in Γ;
- ($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

Avoid redundant applications of the rules!

Sources of non-termination

$$\Box_R\,\dfrac{}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}$$

$$\Diamond_R\,\dfrac{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p}$$

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$$\Box_R\,\dfrac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}$$

$$\vee_R\,\dfrac{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}$$

Limit applications of $\Box_R$ and $\Diamond_L$

$$\Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh} \qquad \Diamond_L \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

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Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

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($\Box_R$) If $x{:}\Box A$ occurs in Δ, then either
a) there is a k such that kRx occurs in $\mathcal{R}$ and $k \sim x$; otherwise
b) there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

If *a*) holds, we say that x is a copy of k at $\mathcal{S}$

A terminating proof search algorithm for labK4

Is $x{:}\Gamma \Rightarrow x{:}A$ derivable in labK4?

0. Place $\mathcal{S}_0 = x{:}\Gamma \Rightarrow x{:}A$ at the root of $\mathcal{T}$.
1. For every topmost labelled sequent $\mathcal{S}_i$ of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: tr, $\wedge_\mathrm{L}, \wedge_\mathrm{R}, \vee_\mathrm{L}, \vee_\mathrm{R}, \rightarrow_\mathrm{L}, \rightarrow_\mathrm{R}, \Box_\mathrm{L}, \Diamond_\mathrm{R}$.
2. If every topmost labelled sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is derivable in labK4.
3. Otherwise, pick a non-initial topmost labelled sequent $\mathcal{S}_k$ of $\mathcal{T}$.
   a) If there are non-redundant $\Box_\mathrm{R}$- or $\Diamond_\mathrm{L}$- rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is not derivable in labK4.

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Theorem (Termination). Root-first proof search in labK4^c terminates in a finite number of steps.

Validity of sequents

Given a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$, and a model $\mathcal{M} = \langle W, R, v \rangle$, let $\text{Lb}(\mathcal{S}) = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \text{Lb}(\mathcal{S}) \rightarrow W$ (interpretation).

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Satisfiability of labelled formulas at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash xRy \quad \textit{iff} \quad \rho(x) R \rho(y)$$

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Satisfiability of sequents at $\mathcal{M}$ under ρ (φ is xRy or $x{:}A$):

$$\mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff}$$

if for all $\varphi \in \mathcal{R} \cup \Gamma$ it holds that $\mathcal{M}, \rho \Vdash \varphi$,

then for some $x{:}D \in \Delta$ it holds that $\mathcal{M}, \rho \Vdash x{:}D$.

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A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ has a countermodel iff there are $\mathcal{M}$, ρ such that:

- $\mathcal{M}, \rho \models \varphi$, for all $\varphi \in \mathcal{R} \cup \Gamma$, and
- $\mathcal{M}, \rho \not\models x{:}D$, for all $x{:}D \in \Delta$.

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Validity of sequents in a class of frames $\mathcal{X}$:

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X},\ \mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta$$

Constructing a countermodel

Lemma. If proof search terminates in step 3, then $\mathcal{S}_0$ has a countermodel.

Proof. Consider $\mathcal{S}_k$, the non-initial topmost nested sequent where the algorithm stopped. We define the model $\mathcal{M}^\times = \langle W^\times, R^\times, v^\times \rangle$ as follows:

- $W^\times = \{x \mid x \text{ occurs in } \mathcal{S}\}$;
- To define $R^\times$, first define:
  - $xR_1^\times y$ iff xRy occurs in $\mathcal{R}$;
  - $xR_2^\times k$ iff x is a $\Box$-copy (or $\Diamond$-copy) of k.

  $\mathcal{R}^\times$ is the transitive closure of $R_1^\times \cup R_2^\times$.
- $v^\times(x) = \{p \mid x{:}p \text{ occurs in } \Gamma\}$.

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Take $\rho^\times(x) = x$, for each label x occurring in $\mathcal{S}$. Then:

- If $x{:}A$ occurs in Γ, then $\mathcal{M}^\times, \rho^\times \models x{:}A$
- If $x{:}A$ occurs in Δ, then $\mathcal{M}^\times, \rho^\times \not\models x{:}A$

Example

$$\frac{}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p, 2{:}\Box p, 3{:}p, 3{:}\Box p}\ \Box_R$$

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$$\frac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}\ \Box_R$$

$$\frac{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}{\Rightarrow 0{:}\Diamond\Box p \vee \Box\bot}\ \vee_R$$

Example

$$\square_R \dfrac{}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\square p, 1{:}\bot, 1{:}\square p, 2{:}p, 2{:}\square p, 3{:}p, 3{:}\square p}$$

$$\Diamond_R \dfrac{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\square p, 1{:}\bot, 1{:}\square p, 2{:}p, 2{:}\square p, 3{:}p, 3{:}\square p}{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\square p, 1{:}\bot, 1{:}\square p, 2{:}p, 2{:}\square p, 3{:}p}$$

$$\mathrm{tr} \dfrac{0R3, 2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\square p, 1{:}\bot, 1{:}\square p, 2{:}p, 2{:}\square p, 3{:}p}{2R3, 0R2, 1R2, 0R1 \Rightarrow 0{:}\Diamond\square p, 1{:}\bot, 1{:}\square p, 2{:}p, 2{:}\square p, 3{:}p}$$

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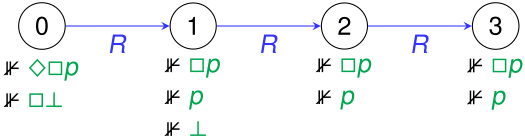
$$\vee_R \dfrac{\Rightarrow 0{:}\Diamond\square p, 0{:}\square\bot}{\Rightarrow 0{:}\Diamond\square p \vee \square\bot}$$



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$$\frac{1R2, 0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p, 2{:}p}{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p}\ \Box_R$$

$$\frac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot, 1{:}\Box p}{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}\ \Diamond_R$$

$$\frac{0R1 \Rightarrow 0{:}\Diamond\Box p, 1{:}\bot}{\Rightarrow 0{:}\Diamond\Box p, 0{:}\Box\bot}\ \Box_R$$

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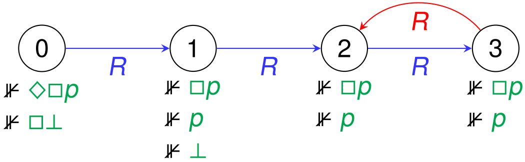

Summing up

	fml. interpr.	invertible rules	analyti-city	termination proof search	counterm. constr.	modu-larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
NK $\cup$ X$^{\diamond}$	yes	yes	yes	yes	yes, easy!	45-clause
labK $\cup$ X	no	yes	yes	yes, for most	yes, easy!	yes

End of content for today's lecture!

Questions?

Exercises

1. Check whether $\Diamond\Box(p \vee \Box(p \rightarrow \bot))$ is valid in K4 using the terminating algorithm for labK4. If the formula is not valid, produce a countermodel.

2. Let $\mathcal{M}^{\times}$ be the countermodel for a labelled sequent $\mathcal{S}$. Verify that $\mathcal{M}^{\times}$ satisfies the frame condition of transitivity.
   Then, for $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{S}$, verify that the Truth Lemma holds, for the cases:
   - If $x{:}\Box A$ occurs in Γ, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}\Box A$
   - If $x{:}\Box A$ occurs in Δ, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}\Box A$