Proof Theory of Modal Logic

Lecture 3
Labelled Proof Systems

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Recap

	fml. interpr.	invertible rules	analyti-city	termination proof search	counterm. constr.	modu-larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$\mathsf{NK} \cup \mathsf{X}^{\Diamond}$	yes	yes	yes	?	?	45-clause

Recap

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G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
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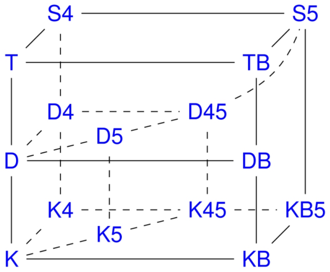
On modularity



$X \subseteq \{d, t, b, 4, 5\}$

$K \cup X$ = one of the logics in the cube

Today's lecture: Labelled Proof Systems

- Labelled sequent calculus for K
- Frame conditions: a general recipe

The labelled approach in the literature

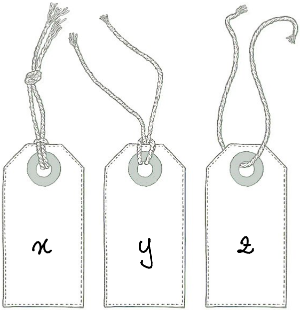
References (non-exhaustive):

- [Kanger, 1957] Spotted formulas for S5
- [Fitting, 1983], [Goré 1998] Tableaux + labels
- [Simpson, 1994], [Viganò, 1998] Natural deduction + labels
- [Mints, 1997], [Viganò, 2000], [Negri ,2005] Sequent calculus + labels

We follow the approach of Negri:

- *Proof analysis in modal logics* [Negri, 2005]
- *Contraction-free sequent calculi for geometric theories with an application to Barr's theorem* [Negri, 2003]

Labelled sequent calculus for K



Enriching the language

$$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

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Labelled formulas

- xRy meaning ‘x has access to y’ (relational atoms)
- $x{:}A$ meaning ‘x satisfies A’

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Labelled sequent

$$\mathcal{R}, \Gamma \Rightarrow \Delta$$

where

- $\mathcal{R}$ is a multiset of relational atoms;
- Γ, Δ are multisets of labelled formulas *without* relational atoms.

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Labelled sequents lack a formula interpretation

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- $x{:}A$ meaning 'x satisfies A'

N.B. This is <u>not</u>
(typed) λ-calculus
$x : A$

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Labelled sequents lack a formula interpretation

Rules of labK

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

Rules of labK

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$$\wedge_{\mathrm{R}}\ \frac{\mathcal{R}, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathrm{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_{\mathrm{L}}\ \frac{\mathcal{R}, x{:}A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathrm{R}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B}$$

$$\to_{\mathrm{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \to B, \Gamma \Rightarrow \Delta} \qquad \to_{\mathrm{R}}\ \frac{x{:}A, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \to B}$$

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$$\Box_L\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

y fresh means $y \neq x$ and y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

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$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

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$$\Diamond_L\ \frac{xRy, \mathcal{R}, y{:}A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \Diamond_R\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A, y{:}A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A}$$

y fresh means $y \neq x$ and y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Provability in labK

We write $\vdash_{\mathrm{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ if there is a derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta$ in labK.

Example: $\vdash_{\mathrm{labK}} \Rightarrow x{:}(\Diamond p \to \Box q) \to \Box(p \to q)$

$$
\to_{\mathrm{R}}\,\dfrac{\Box_{\mathrm{R}}\,\dfrac{\to_{\mathrm{R}}\,\dfrac{\to_{\mathrm{L}}\,\dfrac{\Diamond_{\mathrm{R}}\,\dfrac{\mathrm{init}\,\dfrac{}{xRy, y{:}p \Rightarrow y{:}q, x{:}\Diamond p, y{:}p}}{xRy, y{:}A \Rightarrow y{:}q, x{:}\Diamond p} \qquad \Box_{\mathrm{L}}\,\dfrac{\mathrm{init}\,\dfrac{}{xRy, x{:}\Box q, y{:}q, y{:}p \Rightarrow y{:}q}}{xRy, x{:}\Box q, y{:}p \Rightarrow y{:}q}}{xRy, x{:}\Diamond p \to \Box q, y{:}p \Rightarrow y{:}q}}{xRy, x{:}\Diamond p \to \Box q \Rightarrow y{:}p \to q}}{x{:}\Diamond p \to \Box q \Rightarrow x{:}\Box(p \to q)}}{\Rightarrow x{:}(\Diamond p \to \Box q) \to \Box(p \to q)}
$$

Roadmap

HILBERT-STYLE AXIOM SYSTEM

$\Gamma \vdash A$

LOGICAL CONSEQUENCE

$\Gamma \models A$

$\vdash_{\mathrm{labK}} x : \Gamma \Rightarrow x : A$

LABELLED SEQUENT CALCULUS (without cut)

$x : \Gamma = \{x : G \mid G \in \Gamma\}$

Validity of sequents

Given a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$, and a model $\mathcal{M} = \langle W, R, v \rangle$, let $\mathrm{Lb}(\mathcal{S}) = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \mathrm{Lb}(\mathcal{S}) \to W$ (interpretation).

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Satisfiability of labelled formulas at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash xRy \quad \textit{iff} \quad \rho(x) R \rho(y)$$

$$\mathcal{M}, \rho \Vdash x{:}A \quad \textit{iff} \quad \mathcal{M}, \rho(x) \Vdash A$$

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Satisfiability of sequents at $\mathcal{M}$ under ρ (φ is xRy or $x{:}A$):

$$\mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff}$$

if for all $\varphi \in \mathcal{R} \cup \Gamma$ it holds that $\mathcal{M}, \rho \Vdash \varphi$,

then for some $x{:}D \in \Delta$ it holds that $\mathcal{M}, \rho \Vdash x{:}D$.

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A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ has a countermodel iff there are $\mathcal{M}$, ρ such that:

- $\mathcal{M}, \rho \models \varphi$, for all $\varphi \in \mathcal{R} \cup \Gamma$, and
- $\mathcal{M}, \rho \not\models x{:}D$, for all $x{:}D \in \Delta$.

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Validity of sequents in a class of frames $\mathcal{X}$:

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X},\ \mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta$$

Soundness of labK [Negri, 2009]

Theorem (Soundness). If $\vdash_{\mathsf{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models \mathcal{R}, \Gamma \Rightarrow \Delta$

Roadmap

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$\Gamma \vdash A \quad \Longleftrightarrow \quad \Gamma \models A$$

$\Uparrow$ Sound.

$$\vdash_{labK} x:\Gamma \Rightarrow x:A$$

LABELLED SEQUENT CALCULUS (without cut)

$$x:\Gamma = \{x:G \mid G \in \Gamma\}$$

Towards cut-admissibility of labK 1/2 [Negri, 2005]

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Substitution on labelled formulas:

$$xRy[z/y] \quad := \quad xRz$$

$$y{:}A[z/y] \quad := \quad z{:}A$$

Substitution on multisets of labelled formulas $\Gamma[z/y]$

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Lemma (Substitution). Rule subst is hp-admissible in labK.

$$\text{subst}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}[y/x], \Gamma[y/x] \Rightarrow \Delta[y/x]}$$

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Lemma (Weakening). Rules $\mathsf{wk_L}$, $\mathsf{wk_R}$ are hp-admissible (φ is xRy or $x{:}A$).

$$\mathsf{wk_L}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\varphi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{wk_R}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi}$$

Towards cut-admissibility of labK 2/2 [Negri, 2005]

Lemma (Invertibility).
For every rule r, if the conclusion of r is derivable with a derivation of height *h*, then each of its premisses is derivable, with at most the same *h*.

Rules with variable condition:

$$\frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ \Box_R$$

If $\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A$ is derivable (with derivation height at most n), then for every label $y \neq x$ which does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$, we have that $xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A$ is derivable (with derivation height at most n).

Lemma (Contraction). Rules $\mathsf{ctr_L}$, $\mathsf{ctr_R}$ are hp-admissible (φ is xRy or $x{:}A$).

$$\mathsf{ctr_L}\ \frac{\varphi, \varphi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\varphi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_R}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi}$$

Cut admissibility of labK [Negri, 2005]

Lemma (Cut). The cut rule is admissible.

$$\text{cut}\,\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(c(A), h_1 + h_2)$.

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Proof. By induction on $(c(A), h_1 + h_2)$.

$$\dfrac{\dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ \Box_R \qquad \dfrac{xRz, \mathcal{R}', x{:}\Box A, z{:}A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', x{:}\Box A, \Gamma' \Rightarrow \Delta'}\ \Box_L}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{cut}$$

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$$\dfrac{\dfrac{xRz, \mathcal{R}, \Gamma \Rightarrow \Delta, z{:}A \qquad \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A \qquad xRz, \mathcal{R}', x{:}\Box A, z{:}A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}, \mathcal{R}', z{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{cut}}{\mathcal{R}, \mathcal{R}, xRz, xRz, \mathcal{R}', \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\ \text{cut}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{ctr}_L, \text{ctr}_R$$

Cut admissibility of labK [Negri, 2005]

Lemma (Cut). The cut rule is admissible.

$$\text{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(c(A), h_1 + h_2)$.

$$\text{cut}\ \frac{\Box_R\ \dfrac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A} \qquad \Box_L\ \dfrac{xRz, \mathcal{R}', x{:}\Box A, z{:}A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}', x{:}\Box A, \Gamma' \Rightarrow \Delta'}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$\text{ctr}_L,\text{ctr}_R\ \dfrac{\text{cut}\ \dfrac{xRz, \mathcal{R}, \Gamma \Rightarrow \Delta, z{:}A \qquad \text{cut}\ \dfrac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A \qquad xRz, \mathcal{R}', x{:}\Box A, z{:}A, \Gamma' \Rightarrow \Delta'}{xRz, \mathcal{R}, \mathcal{R}', z{:}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\mathcal{R}, \mathcal{R}, xRz, xRz, \mathcal{R}', \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}}{\mathcal{R}, xRz, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash A$ then $\vdash_{\text{labK}}\ x{:}\Gamma \Rightarrow x{:}A$.

Roadmap

HILBERT-STYLE AXIOM SYSTEM

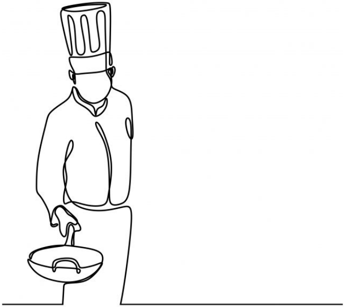
LOGICAL CONSEQUENCE

$$\begin{tikzcd}
\Gamma \vdash A \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow, "\text{compl. (via cut-adm)}"'] & \Gamma \models A \\
& \vdash_{\mathsf{labK}} x:\Gamma \Rightarrow x:A \arrow[u, Rightarrow, "\text{sound.}"']
\end{tikzcd}$$

LABELLED SEQUENT CALCULUS (without cut)

$$x:\Gamma = \{x:G \mid G \in \Gamma\}$$

Frame conditions: a general recipe



Recap: modal logics in the S5-cube

Let $\mathcal{H}\mathsf{K} = \mathcal{H}_{\mathsf{cp}} \cup \{\mathsf{k}, \mathsf{dual}, \mathsf{nec}\}$. Logic K is characterised by the class of all Kripke frames.

Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box\Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \to xRz)$
5	$\Diamond A \to \Box\Diamond A$	Euclideaness	$\forall x \forall y \forall z ((xRy \wedge xRz) \to yRz)$

Take $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$.

We write $\Gamma \vdash_{\mathsf{X}} A$ iff A is derivable from Γ in the axiom system $\mathcal{H}\mathsf{K} \cup \mathsf{X}$.

We denote by $\mathcal{X}$ the class of frames satisfying properties in X.

We write $\Gamma \models_{\mathcal{X}} A$ iff A is logical consequence of Γ in the class of frames $\mathcal{X}$.

Theorem. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, $\Gamma \vdash_{\mathsf{X}} A$ iff $\Gamma \models_{\mathcal{X}} A$.

Main ingredients

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Proof systems for geometric theories, [Negri, 2003]:

"axioms-as-rules"

How to transform axioms of geometric theories (geometric implications) into rules, preserving the structural properties of the calculus.

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The first-order logic formulas corresponding to the frame conditions above (and many more!) are geometric implications

Main ingredients (once more)

1. "axioms-as-rules" method [Negri, 2003] for FOL
   geometric axioms can be turned into sequent calculus rules
   (general method to define cut-free sequent calculi for geometric theories)

2. Frame conditions, read as FOL formulas, are geometric axioms

3. We can define cut-free labelled sequent calculi for modal logics whose frame conditions can be expressed as geometric axioms [Negri, 2005]

First-order languages

First-order languages

A first-order signature is a tuple $\sigma = \langle c, d, \ldots, f, g, \ldots p, q, \ldots \rangle$

- Constant symbols $c, d, \ldots$
- Function symbols $f, g, \ldots$, each with arity > 0
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A first-order language over a signature σ, denoted $\mathcal{L}(\sigma)$, consists of:

- The terms generated from a countably many variables $x, y, \ldots$ using the constants and function symbols of σ;
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Example.
$\mathcal{L}^{=}(0, suc^1, +^2, \times^2)$ is the language of arithmetic
$\mathcal{L}(R^2)$ is the language we use to express frame conditions

Geometric theories

Fix a first-order language $\mathcal{L}(\sigma)$ (with or without equality).

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A first-order theory over $\mathcal{L}(\sigma)$ is a set of closed formulas of $\mathcal{L}(\sigma)$.

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$$\forall \vec{x}(A \rightarrow B), \quad \text{for } A, B \text{ geometric formulas}$$

A geometric theory over $\mathcal{L}(\sigma)$ is a first-order theory over $\mathcal{L}(\sigma)$ whose formulas are geometric implications.

Example: Peano Arithmetic and Robinson Arithmetic

$$\mathcal{L}^{=}(0, suc, +, \times)$$

Axiomatisation of first-order logic with equality, plus:

$$\left.\begin{array}{ll}
1 & \forall x(0 \neq suc(x)) \\
2 & \forall x \forall y(suc(x) = suc(y) \to x = y) \\
3 & \forall x(x + 0 = x) \\
4 & \forall x \forall y(x + suc(y) = suc(x + y)) \\
5 & \forall x(x \times 0 = 0) \\
6 & \forall x \forall y(x \times suc(y) = (x \times y) + x) \\
7 & x = 0 \vee \exists y(x = suc(y))
\end{array}\right\} \text{Robinson Arithmetic RA}$$

$Ind(A) \quad \big(A(0) \wedge \forall x(A(x) \to A(sucx))\big) \to \forall x A(x)$ for any $A(x)$

Peano Arithmetic : $RA \setminus \{7\} \cup Ind(A)$

From geometric axioms to rules [Negri, 2003]

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x}\Big(P \rightarrow \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big)$$

- $\vec{x}$, $\vec{y}_1, \ldots, \vec{y}_m$ are (possibly empty, disjoint) vectors of variables;
- $m \geq 0$;
- P, $Q_1, \ldots, Q_m$ are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
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Geometric axioms can be turned into sequent calculus rules:

$$\mathrm{GA}\ \frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}$$

- Π is the multiset of atomic formulas in P;
- Ξ_i is the multiset of atomic formulas in Q_i, for each $i \leq m$;
- $\vec{z}_1, \ldots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

From geometric axioms to labelled rules [Negri, 2003]

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\cancel{\mathcal{L}(\sigma)}$ having the form:
$\mathcal{L}(R)$

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Examples

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Labelled calculi for the S5-cube [Negri, 2005]

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \text{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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For $X \subseteq \{d, t, b, 4, 5\}$, $labK \cup X$ is defined by adding to labK the rules for frame conditions corresponding to elements of X, plus the rules obtained to satisfy the closure condition (contracted instances of the rules):

$$\text{euc}\ \frac{yRy, xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \text{euc}'\ \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example: $labK \cup \{5\}$ denotes the proof system $labK \cup \{euc, euc'\}$.

We denote by $\vdash_{labK \cup X} \mathcal{S}$ derivability of labelled sequent $\mathcal{S}$ in $labK \cup X$.

Soundness and completeness of labK ∪ X [Negri, 2005]

For X $\subseteq \{d, t, b, 4, 5\}$:

Theorem (Soundness). If $\vdash_{\mathrm{labK} \cup \mathrm{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$.

Example. If the premiss of rule ser is valid in all serial models, then its conclusion is valid in all serial models.

$$\mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Lemma (Cut). The cut rule is admissible in labK ∪ X:

$$\mathsf{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_{\mathrm{K} \cup \mathrm{X}} A$ then $\vdash_{\mathrm{labK} \cup \mathrm{X}} x{:}\Gamma \Rightarrow x{:}A$.

Roadmap

$X \subseteq \{d, t, b, 4, 5\}$

HILBERT-STYLE AXIOM SYSTEM

$\Gamma \vdash_X A$

LOGICAL CONSEQUENCE

$\Gamma \models_X A$

compl.
(via cut-adm)

sound.

$\vdash_{labK \cup X} x : \Gamma \Rightarrow x : A$

LABELLED SEQUENT CALCULUS (without cut)

$x : \Gamma = \{ x : G \mid G \in \Gamma \}$

Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$\mathrm{NK} \cup \mathrm{X}^{\diamond}$	yes	yes	yes	?	?	45-clause
$\mathrm{labK} \cup X$	no	yes	yes	?	?	yes

Beyond geometric axioms

Beyond geometric axioms

- Systems of rules [Negri, 2016], to capture theories / logics characterized by generalized geometric implications:

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GA_0 &= \forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big) \\
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for $k_1, \ldots, k_m \geq n$

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  - Transitivity: R is transitive
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  [Negri, 2005]: labelled proof system for GL!

Exercises

d	$\Box A \to \Diamond A$
t	$\Box A \to A$
b	$A \to \Box \Diamond A$
4	$\Box A \to \Box \Box A$
5	$\Diamond A \to \Box \Diamond A$

1. For $X \in \{d, t, b, 4, 5\}$, show that the axiom X is derivable in the labelled sequent calculus labK $\cup$ X.
2. Show that the rules ref, tr, sym, ser, euc are sound in the corresponding class of frames.
3. Write down the sequent calculus rules corresponding to the axioms of Robinson Arithmetic. These rules are to be added to the sequent calculus for first-order logic with equalitity, where one can show that cut is eliminable. Can we use the results from [Negri, 2003] to prove consistency of Robinson Arithmetic? If yes, how?