Proof Theory of Modal Logic

Lecture 2
Nested Sequents

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Recap

Derivable rule vs. admissible rules

$$R \; \frac{P_1 \quad .. \quad P_n}{c}$$

sc : sequent calculus (set of rules)

R is derivable : there is a derivation of c from the premisses $P_1, .., P_n$ in sc

R is admissible : if the premisses $P_1, .., P_n$ are derivable, then the conclusion c is derivable in sc.

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Example : Rule $\wedge 3$ derivable in G3cp

$$\frac{A, B, C, \Gamma \Rightarrow \Delta}{(A \wedge B) \wedge C, \Gamma \Rightarrow \Delta} \wedge 3 \qquad \frac{\dfrac{A, B, C, \Gamma \Rightarrow \Delta}{A \wedge B, C, \Gamma \Rightarrow \Delta} \wedge_L}{(A \wedge B) \wedge C, \Gamma \Rightarrow \Delta} \wedge_L$$

R is <u>admissible</u> : <u>if</u> the premisses $P_1, .., P_n$ are derivable, <u>then</u> the conclusion c is derivable in sc.

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Derivable rule vs. admissible rules

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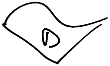
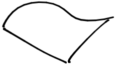
R is <u>admissible</u> : if the premisses $P_1, \dots, P_n$ are derivable, then the conclusion C is derivable in SC.

Example : wk_L ~~derivable~~ admissible in G3cp

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} wk_L$$



if $\Gamma \Rightarrow \Delta$ then



$A, \Gamma \Rightarrow \Delta$

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Recap

Consistency of PA (Peano Arithmetic)

- Language of arithmetic $\{0, s, +, \times\}$
- PA is a set of axioms and inference rules $\vdash_{PA} A$

$$\mathrm{FOL}^{=} + \left\{ \begin{array}{l} \forall x\,(0 \neq s(x)) \\ \forall x \forall y\,(s(x) = s(y) \to x = y) \\ \vdots \end{array} \right\} + \text{induction rule}$$

- Define a sequent calculus sound and complete w.r.t. PA
  $\vdash_{SC} \Gamma \Rightarrow \Delta$ with cut
- Prove cut elimination for SC
- Observe that cut-free proofs are analytic: every formula occurring in them is a subformula of formulas in the conclusion
- Consistency of PA: suppose $\vdash_{PA} \bot$. Then, by completeness, $\vdash_{SC} \Rightarrow \bot$.
  By cut-elimination, $\vdash_{SC \setminus \{cut\}} \Rightarrow \bot$. contradiction.

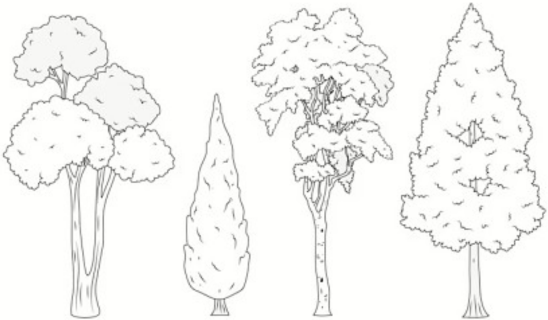
Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no

Today's lecture: Nested Sequents

- Nested sequents for K
- Nested sequents for the S5-cube

Nested sequents for K



Nested sequents in the literature

Independently introduced in:

- [Bull, 1992]; [Kashima, 1994] $\rightsquigarrow$ *nested sequents*
- [Brünnler, 2006], [Brünnler, 2009] $\rightsquigarrow$ *deep sequents*
- [Poggiolesi, 2008], [Poggiolesi, 2010] $\rightsquigarrow$ *tree-hypersequents*

Main references for this lecture:

- [Lellmann & Poggiolesi, 2022 (arXiv)]
- [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- [Marin & Straßburger, 2014]

One-sided sequents

Sequent $\quad i(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta \quad \Gamma, \Delta$ multisets of formulas

One-sided sequents

Sequent	$\Gamma \Rightarrow \Delta$	Γ, Δ multisets of formulas
One-sided sequent	Γ	Γ multiset of formulas

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Rules of G3cp^{one}

$$\text{init}\,\frac{}{\Gamma, p, \overline{p}} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

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Exercise. $\vdash_{\mathrm{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathrm{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

Nested sequents for modal logic

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- If Γ is a nested sequent, then $[\Gamma]$ is a nested sequent.
  We call $[\Gamma]$ a boxed sequent.

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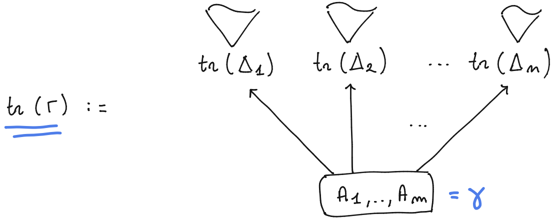
Nested sequents are multisets of formulas and boxed sequents:

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

Trees

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \ldots$ are multisets of formulas:



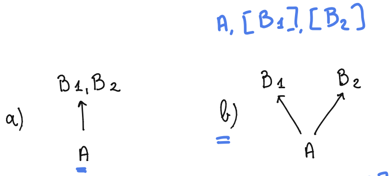
The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

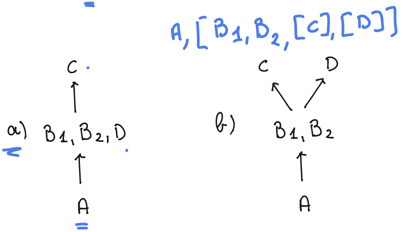
1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?



2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?



3) $\Gamma = A, [B_1, B_2], [C, [D]]$

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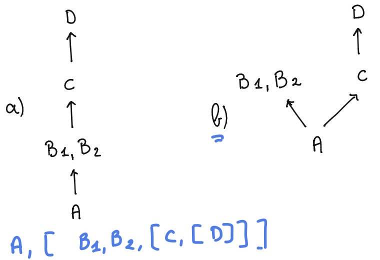

Contexts

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

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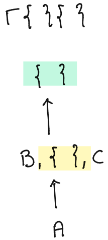
- Binary context $\Gamma\{\,\}\{\,\}$ $\rightsquigarrow$ $\Gamma\{\Delta_1\}\{\Delta_2\}$: filling $\Gamma\{\,\}\{\,\}$ with Δ_1, Δ_2

$\Gamma\{\,\}\{\,\} = A, [B, \{\,\}, [\{\,\}], C]$



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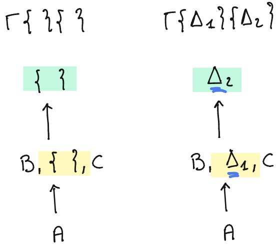
Δ_1, Δ_2 nested sequents

$\Gamma\{\Delta_1\}\{\Delta_2\} = \underline{A, [B, \Delta_1, [\Delta_2], C]}$



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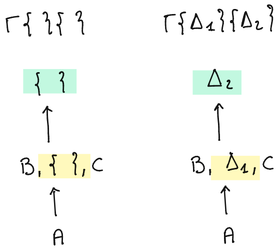

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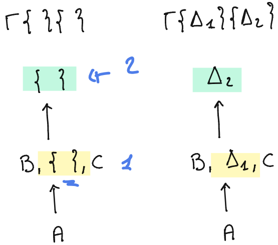

The depth $depth(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $depth(\{\,\}) := 0$;
- $depth(\Gamma\{\,\}, \Delta) := depth(\Gamma\{\,\})$;
- $depth([\Gamma\{\,\}]) := depth(\Gamma\{\,\}) + 1$.

$depth(\Gamma\{\,\}\{\Delta_1\}) = 1$

$depth(\Gamma\{\Delta_1\}\{\,\}) = 2$

Rules of NK

$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Rules of NK

nK

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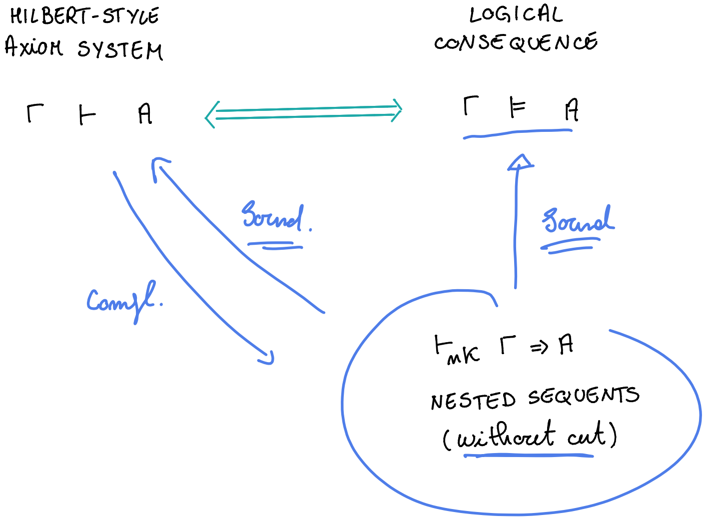
Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

$$\vee\ \dfrac{\Box\ \dfrac{\vee\ \dfrac{\wedge\ \dfrac{\Diamond\ \dfrac{\text{init}\ \dfrac{}{\Diamond p, [p, \bar{p}, q]}}{\Diamond p, [\bar{p}, q]} \qquad \Diamond\ \dfrac{\text{init}\ \dfrac{}{\Diamond \bar{q}, [\bar{q}, \bar{p}, q]}}{\Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p} \vee q]}}{\Diamond p \wedge \Diamond \bar{q}, \Box(\bar{p} \vee q)}}{(\Diamond p \wedge \Diamond \bar{q}) \vee \Box(\bar{p} \vee q)}$$

Roadmap


HILBERT-STYLE
AXIOM SYSTEM
$\Gamma \vdash A$
LOGICAL
CONSEQUENCE
$\Gamma \models A$
Sound.
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NESTED SEQUENTS
(without cut)

Validity of nested sequents [Kuznets & Straßburger, 2018]

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

$$\begin{array}{ccc} \delta & \quad & f(\delta) \\ \uparrow & & \uparrow R \\ \gamma & & f(\gamma) \end{array}$$

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A nested sequent Γ is satisfied by an $\mathcal{M}$-map for Γ iff

$$\mathcal{M}, f(\delta) \models B, \text{ for some } \underline{\delta \in tr(\Gamma)}, \text{ for some } \underline{B \in \delta}$$

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A nested sequent is valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$.

Soundness of NK

Lemma. If Γ is derivable in NK then Γ is valid in all Kripke frames.

Proof. Induction on height of derivation of Γ.

Case $\Box$: The last rule applied in deriv. of Γ is $\Box$

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\Box$$

need to prove:
if $\Gamma\{[A]\}$ is valid, then $\Gamma\{\Box A\}$ is valid

contrapositive: if $\Gamma\{\Box A\}$ is refuted, then $\Gamma\{[A]\}$ is refuted

$\Downarrow$

there is $\mathcal{M}$, f s.t.

$\Rightarrow$ $\mathcal{M}, f(\delta) \not\models B$ for all $\delta \in tr(\Gamma\{\Box A\})$, for all $B \in \delta$

Let γ be such that $\Box A \in \gamma \in tr(\Gamma\{\Box A\})$

$\mathcal{M}, f(\gamma) \not\models \Box A$ $\rightsquigarrow$ there is $w \in W$ s.t. $f(\gamma) R w$
$\mathcal{M}, w \not\models A$

Take $\mathcal{M}$, take f' s.t. $f'(\varepsilon) = w$ and $f'(\delta) = f(\delta)$ for all other $\delta \in tr(\Gamma\{[A]\})$.

Check : $\mathcal{M}, f'(\delta) \not\models B$ for all $\delta \in tr(\Gamma\{[A]\})$.

Roadmap

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$
\begin{tikzcd}
\Gamma \vdash A \arrow[r, Leftrightarrow] \arrow[dr, bend right] & \Gamma \models A \\
& \vdash_{mK} \Gamma \Rightarrow A \arrow[u, Rightarrow, "\text{Sound.}"']
\end{tikzcd}
$$

NESTED SEQUENTS

(without cut)

Completeness of NK

$$\text{wk}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \text{ctr}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

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Lemma. The rules wk and ctr are hp-admissible in NK.

Completeness of NK

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Lemma. The rules wk and ctr are hp-admissible in NK.

Lemma. All the rules of NK are hp-invertible.

Completeness of NK

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Lemma. The rules wk and ctr are hp-admissible in NK.

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Theorem. The cut rule is admissible in NK.

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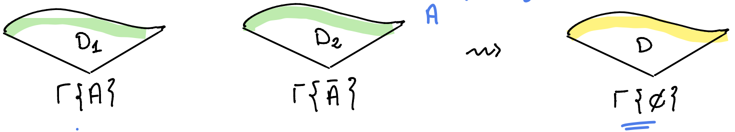
Lemma. The rules wk and ctr are hp-admissible in NK.

Lemma. All the rules of NK are hp-invertible.

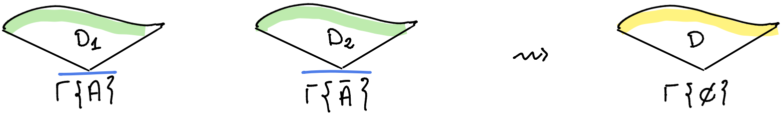
Theorem. The cut rule is admissible in NK.

Proof sketch. Assume that the two premisses of cut are derivable in NK, and show how to construct a derivation of the conclusion of the conclusion. Lexicographic induction on (c, h). sum of heights of D_1 and D_2

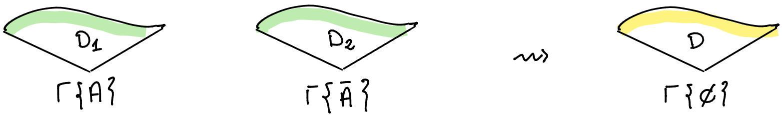
complexity of A



One cut reduction case



One cut reduction case



$A := \Box B$, and $\Box B$ is principal in the last rule applied in $\mathcal{D}_1, \mathcal{D}_2$

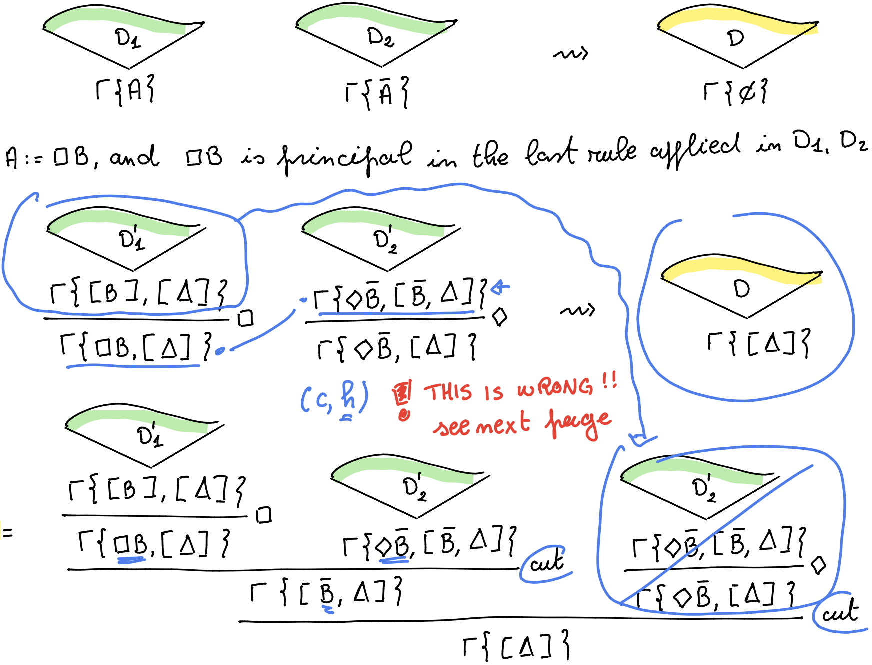
One cut reduction case

$$\dfrac{\mathcal{D}_1}{\Gamma\{A\}} \qquad \dfrac{\mathcal{D}_2}{\Gamma\{\bar{A}\}} \qquad \rightsquigarrow \qquad \dfrac{\mathcal{D}}{\Gamma\{\emptyset\}}$$

$A := \Box B$, and $\Box B$ is principal in the last rule applied in $\mathcal{D}_1, \mathcal{D}_2$

$$\dfrac{\dfrac{\mathcal{D}_1'}{\Gamma\{[B],[\Delta]\}}}{\Gamma\{\Box B,[\Delta]\}}\,\Box \qquad \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma\{\Diamond\bar{B},[\bar{B},\Delta]\}}}{\Gamma\{\Diamond\bar{B},[\Delta]\}}\,\Diamond \qquad \rightsquigarrow \qquad \dfrac{\mathcal{D}}{\Gamma\{[\Delta]\}}$$

One cut reduction case



$$\begin{matrix}\mathcal{D}_1\\ \Gamma\{A\}\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma\{\bar A\}\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}\\ \Gamma\{\emptyset\}\end{matrix}$$

$A := \Box B$, and $\Box B$ is principal in the last rule applied in $\mathcal{D}_1, \mathcal{D}_2$

$$\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma\{[B],[\Delta]\}\end{matrix}}{\Gamma\{\Box B,[\Delta]\}}\,\Box \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \Gamma\{\Diamond\bar B,[\bar B,\Delta]\}\end{matrix}}{\Gamma\{\Diamond\bar B,[\Delta]\}}\,\Diamond \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}\\ \Gamma\{[\Delta]\}\end{matrix}$$

(c, h)

THIS IS WRONG !!
see next page

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma\{[B],[\Delta]\}\end{matrix}}{\Gamma\{\Box B,[\Delta]\}}\,\Box \qquad \begin{matrix}\mathcal{D}_2'\\ \Gamma\{\Diamond\bar B,[\bar B,\Delta]\}\end{matrix}}{\Gamma\{[\bar B,\Delta]\}}\,\mathsf{cut} \qquad \cancel{\dfrac{\begin{matrix}\mathcal{D}_2'\\ \Gamma\{\Diamond\bar B,[\bar B,\Delta]\}\end{matrix}}{\Gamma\{\Diamond\bar B,[\Delta]\}}\,\Diamond}}{\Gamma\{[\Delta]\}}\,\mathsf{cut}$$

One cut reduction case

$A := \Box B$, and $\Box B$ is principal in the last rule applied in $\mathcal{D}_1, \mathcal{D}_2$

$$\dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma\{[B],[\Delta]\}\end{matrix}}{\Gamma\{\Box B,[\Delta]\}}\,\Box \qquad \dfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma\{\Diamond\bar{B},[\bar{B},\Delta]\}\end{matrix}}{\Gamma\{\Diamond\bar{B},[\Delta]\}}\,\Diamond \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D} \\ \Gamma\{[\Delta]\}\end{matrix}$$

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma\{[B],[\Delta]\}\end{matrix}}{\Gamma\{[B,\Delta],[\Delta]\}}\,wk \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma\{[B],[\Delta]\}\end{matrix}}{\Gamma\{\Box B,[\Delta]\}}\,\Box}{\Gamma\{\Box B,[\bar{B},\Delta]\}}\,wk \qquad \begin{matrix}\mathcal{D}_2' \\ \Gamma\{\Diamond\bar{B},[\bar{B},\Delta]\}\end{matrix}}{\Gamma\{[\bar{B},\Delta]\}}\,cut}{\Gamma\{[\bar{B},\Delta],[\Delta]\}}\,wk}{\Gamma\{[\Delta],[\Delta]\}}\,cut}{\Gamma\{[\Delta]\}}\,ctr$$

Roadmap

Theorem. If $\Gamma \vdash A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$
\begin{tikzcd}
\Gamma \vdash A \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow, "\text{compl. (via cut-adm)}"'] & \Gamma \models A \arrow[d, bend right] \\
 & \vdash_{NK} \Gamma \Rightarrow A \arrow[u, Rightarrow, bend right, "\text{Sound.}"']
\end{tikzcd}
$$

NESTED SEQUENTS

(without cut)

Semantic completeness (tomorrow)

Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

Theorem (Semantic Completeness). If $\Gamma \models A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.

Proof or countermodel (tomorrow)

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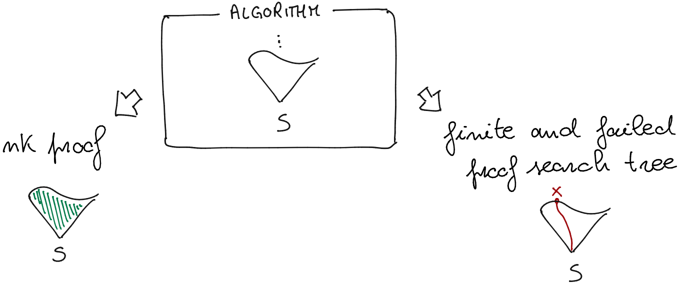
Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

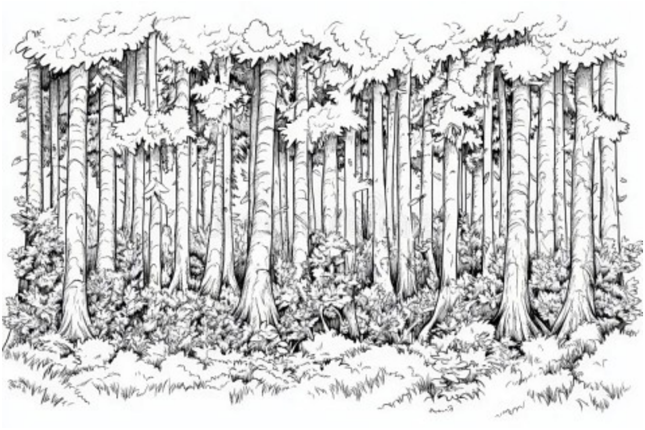
Proof (sketch). Algorithm implementing proof search in nK



Example (tomorrow)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{init}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{p}, p], \Box q$}
\AxiomC{$\Diamond(\bar{p} \land \bar{q}), [\bar{q}, p], [\bar{p}, q]$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{q}, p], [\bar{q}, q]$}
\LeftLabel{$\land$}
\BinaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{q}, p], [\bar{p} \land \bar{q}, q]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{q}, p], [q]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{q}, p], \Box q$}
\LeftLabel{$\land$}
\BinaryInfC{$\Diamond(\bar{p} \land \bar{q}), [\bar{p} \land \bar{q}, p], \Box q$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), [p], \Box q$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), \Box p, \Box q$}
\LeftLabel{$\lor$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}), \Box p \lor \Box q$}
\LeftLabel{$\lor$}
\UnaryInfC{$\Diamond(\bar{p} \land \bar{q}) \lor (\Box p \lor \Box q)$}
\end{prooftree}
$$

Nested sequents for the S5-cube



Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^{\diamond}$

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta, \diamond A], A\}}{\Gamma\{[\Delta, \diamond A]\}}$$

$$4^{\diamond}\ \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\Gamma\{\diamond A, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

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For $\mathrm{X} \subseteq \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, we write $\mathrm{X}^{\diamond}$ for the corresponding subset of $\{\mathrm{d}^{\diamond}, \mathrm{t}^{\diamond}, \mathrm{b}^{\diamond}, 4^{\diamond}, 5^{\diamond}\}$. We shall consider the calculi $\mathrm{NK} \cup \mathrm{X}^{\diamond}$.

Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^\Diamond$

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Example. Proof of $\Box p \to \Box\Box p$ in $\mathsf{NK} \cup \{\mathsf{t}, 4\}$

$$\vee \dfrac{\Box \dfrac{\Box \dfrac{4^\Diamond \dfrac{4^\Diamond \dfrac{\mathsf{t}^\Diamond \dfrac{\mathsf{init}\ \overline{\quad\quad}}{\Diamond \bar{p}, [\Diamond \bar{p}, [\Diamond \bar{p}, \bar{p}, p]]}}{\Diamond \bar{p}, [\Diamond \bar{p}, [\Diamond \bar{p}, p]]}}{\Diamond \bar{p}, [\Diamond \bar{p}, [p]]}}{\Diamond \bar{p}, [[p]]}}{\Diamond \bar{p}, [\Box p]}}{\Diamond \bar{p}, \Box\Box p}}{\Diamond \bar{p} \vee \Box\Box p}$$

Structural rules [Brünnler, 2009]

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For $X \subseteq \{d, t, b, 4, 5\}$:

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Proposition. Rule $5^\Diamond$ is derivable in $\mathsf{NK} \cup \{5_1^\Diamond, 5_2^\Diamond, 5_3^\Diamond\} \cup \{\mathsf{ctr}\}$.

$$5^\Diamond \ \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}} \ depth(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$5_1^\Diamond \ \frac{\Gamma\{[\Delta, \Diamond A], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 5_2^\Diamond \ \frac{\Gamma\{[\Delta, \Diamond A], [\Lambda, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Lambda]\}} \qquad 5_3^\Diamond \ \frac{[\Delta, \Diamond A, [\Lambda, \Diamond A]]}{\Gamma\{[\Delta, \Diamond A, [\Lambda]]\}}$$

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For $X \subseteq \{d, t, b, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Theorem. If Γ is derivable in $NK \cup X^{\diamond}$ then Γ is valid in all X-frames.

Three problems for completeness

Three problems for completeness

- Axiom 5, that is, $\Diamond A \to \Box\Diamond A$, is valid in all $\{\mathsf{b}, 4\}$-frames, but it is not derivable in $\mathsf{NK} \cup \{\mathsf{b}^{\Diamond}, 4^{\Diamond}\}$.

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Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$

$$
\vee\,\dfrac{\Box\,\dfrac{\Box\,\dfrac{\mathrm{b}^{\Diamond}\,\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}}{\Box\bar{p}, [\Diamond p]}}{\Box\bar{p}, \Box\Diamond p}}{\Box\bar{p} \vee \Box\Diamond p}
$$

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- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{\mathrm{t}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{t}^{\Diamond}, 5^{\Diamond}\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\,\mathrm{b}^{\Diamond}}{\Box\bar{p}, [\Diamond p]}\,\Box}{\Box\bar{p}, \Box\Diamond p}\,\Box}{\Box\bar{p} \vee \Box\Diamond p}\,\vee$$

Three problems for completeness

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Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{b^\Diamond, 4^\Diamond\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\, b^\Diamond}{\Box\bar{p}, [\Diamond p]}\, \Box}{\Box\bar{p}, \Box\Diamond p}\, \Box}{\Box\bar{p} \vee \Box\Diamond p}\, \vee
$$

Solution # 1 [Brünnler, 2009]

For each set of frames characterised by the 5-axioms, there is at least one combination of modal rules which is complete.

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For each set of frames characterised by the 5-axioms, there is at least one combination of modal rules which is complete.

For $X \subseteq \{d, t, b, 4, 5\}$, the 45-closure of X is defined as:

$$\hat{X} = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$

We say that X is 45-closed if $X = \hat{X}$.

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Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

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For each set of frames characterised by the 5-axioms, there is at least one combination of modal rules which is complete.

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We say that X is 45-closed if $X = \hat{X}$.

Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

To prove:

semantic

Theorem (Completeness). For $\underline{X \subseteq \{d, t, b, 4, 5\}}$, if Γ is X-valid, then Γ is derivable in $NK \cup \hat{X}^{\diamond}$.

Solution # 1 - Syntactic completeness [Brünnler, 2009]

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\diamond}$.

Solution # 1 - Syntactic completeness [Brünnler, 2009]

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\Diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\Diamond}$.

The proof uses:

- A generalised version of cut (Y-cut, eliminable)

$$\text{cut}\ \dfrac{\Box\ \dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}$$

- Additional structural modal rules (admissible)

Solution # 1 - Syntactic completeness [Brünnler, 2009]

Theorem (Cut-elimination). For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ 45-closed, if Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^\Diamond \cup \{\mathsf{cut}\}$, then it is derivable in $\mathsf{NK} \cup \mathsf{X}^\Diamond$.

The proof uses:

- A generalised version of cut (Y-cut, eliminable)

$$\mathsf{cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad \mathsf{tr}^\Diamond\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma\{\Box A,[\Delta]\}}{\Gamma\{\Box A,[\Box A,\Delta]\}}\,wk \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Box A,[\Delta]\}}\,cut$$

- Additional structural modal rules (admissible)

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cut and Y-cut

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

In the Y-cut:

- $\{\Delta\}^n$ denotes $\overbrace{\{\Delta\}\ldots\{\Delta\}}^{n \text{ times}}$;
- $n \geq 0$;
- $\mathsf{Y} \subseteq \{4,5\}$;
- there is a derivation of $\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n$ to $\Gamma\{\Diamond\overline{A}\}\{\emptyset\}^n$ in system $\mathsf{Y}^{\Diamond}$.

Example: 4-cut

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

If $\mathrm{Y} = \{4\}$, then $\Gamma\{\ \}\{\ \}^n$ is of the form $\Gamma_1\{\{\ \}, \Gamma_2\{\ \}^n\}$:

$$\text{4-cut}\ \frac{\Gamma_1\{\{\Box A\}, \Gamma_2\{\emptyset\}^n\} \quad \Gamma_1\{\{\Diamond A\}, \Gamma_2\{\Diamond A\}^n\}}{\Gamma_1\{\{\emptyset\}, \Gamma_2\{\emptyset\}^n\}}$$

$$\text{cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \qquad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{\Diamond\overline{A}, [\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \text{4-cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \qquad \Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{[\Delta]\}}$$

$\Gamma_2\{\ \}^1 = [\{\ \}, \Delta]$

$\Gamma_1\{\ \}\{\ \} = \Gamma\{\{\ \}, [\{\ \}, \Delta]\}$

Structural modal rules

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

$$\mathsf{4}^{[\,]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad \mathsf{5}^{[\,]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathit{depth}(\Gamma\{\ \}\{[\Delta]\}) > 0$$

For $X \subseteq \{d, t, b, 4, 5\}$, we write $X^{[\,]}$ for the corresponding subset of $\{d^{[\,]}, t^{[\,]}, b^{[\,]}, 4^{[\,]}, 5^{[\,]}\}$.

Structural modal rules

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

$$4^{[\,]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad 5^{[\,]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathit{depth}(\Gamma\{\ \}\{[\Delta]\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},4,5\}$, we write $\mathsf{X}^{[\,]}$ for the corresponding subset of $\{\mathsf{d}^{[\,]},\mathsf{t}^{[\,]},\mathsf{b}^{[\,]},4^{[\,]},5^{[\,]}\}$.

Example. Proof of $\Diamond A \to \Box\Diamond A$ in $\mathsf{NK} \cup \{\mathsf{b}^{[\,]},4^{[\,]}\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[[[\bar{p},p],\Diamond p]]}\ \mathsf{init}}{[[[\bar{p}],\Diamond p]]}\ \Diamond}{[[[\bar{p}]],\Diamond p]}\ 4^{[\,]}}{[\bar{p}],[\Diamond p]}\ \mathsf{b}^{[\,]}}{\Box\bar{p},[\Diamond p]}\ \Box}{\Box\bar{p},\Box\Diamond p}\ \Box}{\Box\bar{p}\vee\Box\Diamond p}\ \vee$$

Cut-admissibility

Theorem (Cut-admissibility). For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ 45-closed, the cut rule and the Y-cut rule are admissible in $\mathsf{NK} \cup \mathsf{X}^\Diamond$.

$$\mathsf{cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad 4^\Diamond\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}} \quad\rightsquigarrow\quad \mathsf{4\text{-}cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{[\Delta]\}}$$

$$\mathsf{4\text{-}cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\Box A,[[\Sigma]]\}} \qquad \Diamond\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}} \quad\rightsquigarrow$$

$$\rightsquigarrow\quad \mathsf{cut}\,\dfrac{4^{[\,]}\,\dfrac{4^{[\,]}\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{[[A],[\Sigma]]\}}}{\Gamma\{[[A,\Sigma]]\}} \qquad \mathsf{4\text{-}cut}\,\dfrac{\mathsf{wk}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\Box A,[[\Sigma]]\}}}{\Gamma\{\Box A,[[\overline{A},\Sigma]]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{[[\overline{A},\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}$$

Roadmap

$X \subseteq \{d, t, b, 4, 5\}$ and X 45-closed:

HILBERT-STYLE AXIOM SYSTEM

LOGICAL CONSEQUENCE

$$
\begin{tikzcd}
\Gamma \vdash_X A \arrow[rr, Leftrightarrow] \arrow[drr, Rightarrow, "\text{compl. (via cut-adm)}"'] & & \Gamma \models_X A \\
& & \vdash_{mK \cup X^\Diamond} \Gamma \Rightarrow A \arrow[u, Rightarrow, "\text{sound.}"']
\end{tikzcd}
$$

NESTED SEQUENTS

(without cut)

Solution # 2 [Marin & Straßburger, 2014]

Can we get rid of the 45-closure condition?

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YES: by adding to NK both the propagation rules $X^{\diamond}$ and the structural rules $X^{[]}$. The price to pay is that contraction is no longer admissible.

Theorem. For $X = \{d, t, b, 4, 5\}$, and Γ a set of formulas, it holds that
Γ is derivable in $NK_{ctr} \cup X^{\diamond}_{ctr} \cup X^{[]}$ iff Γ is X-valid.

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NO, some combinations are incomplete, and one example is given in [Marin & Straßburger, 2014].

Summing up

	fml. interpr.	invertible rules	analyti-city	termination proof search	counterm. constr.	modu-larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$NK \cup X^{\diamond}$	yes	yes	yes	yes	yes	45-clause

Beyond nested sequents

Other ‘structured’ approaches to define proof systems for modal logics:

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To get started: [Poggiolesi, 2008], [Lellmann, 2016]

A hypersequent $\mathcal{H}$ is a finite multiset of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid .. \mid \Gamma_n \Rightarrow \Delta_n$$

$$\Box_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta} \qquad t \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

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- .. and many more! For an overview: [Lyon et al., 2025]

End of content for today’s lecture!

Questions?

Exercises

$$
\begin{array}{ll}
\mathrm{d} & \Box A \rightarrow \Diamond A \\
\mathrm{t} & \Box A \rightarrow A \\
\mathrm{b} & A \rightarrow \Box\Diamond A \\
4 & \Box A \rightarrow \Box\Box A \\
5 & \Diamond A \rightarrow \Box\Diamond A
\end{array}
$$

1. For $X \in \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, show that the axiom X is derivable in the nested sequent calculus $\mathrm{NK} \cup \mathrm{X}^{\Diamond}$.
2. Show that axiom 4 is valid in all $\{\mathrm{t}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{t}^{\Diamond}, 5^{\Diamond}\}$. Show that the axiom is derivable in $\mathrm{NK} \cup \{\mathrm{t}^{[]}, 5^{[]}\}$.
3. Show that 4 is valid in all $\{\mathrm{b}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 5^{\Diamond}\}$. Show that the axiom is derivable in $\mathrm{NK} \cup \{\mathrm{b}^{[]}, 5^{[]}\}$.
4. Derive axioms t, b and 5 in the hypersequent calculus for S5.