Proof Theory of Modal Logic

Lecture 2
Nested Sequents

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ILLC, Universtiy of Amsterdam

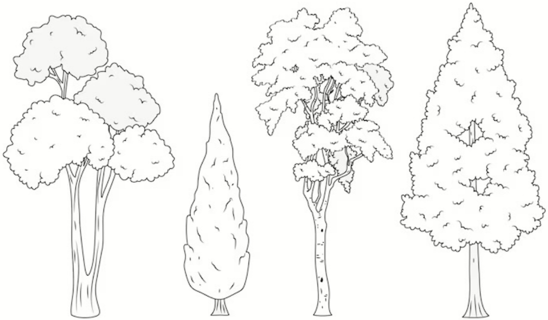
5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Recap

Today's lecture: Nested Sequents

- Nested sequents for K
- Nested sequents for the S5-cube

Nested sequents for K



Nested sequents in the literature

Independently introduced in:

- [Bull, 1992]; [Kashima, 1994] $\rightsquigarrow$ *nested sequents*
- [Brünnler, 2006], [Brünnler, 2009] $\rightsquigarrow$ *deep sequents*
- [Poggiolesi, 2008], [Poggiolesi, 2010] $\rightsquigarrow$ *tree-hypersequents*

Main references for this lecture:

- [Lellmann & Poggiolesi, 2022 (arXiv)]
- [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- [Marin & Straßburger, 2014]

One-sided sequents

Sequent	$\Gamma \Rightarrow \Delta$	Γ, Δ multisets of formulas

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Rules of G3cp^{one}

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Exercise. $\vdash_{\text{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\text{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

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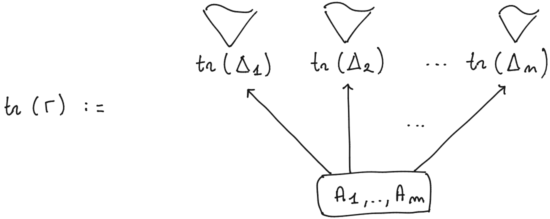
Nested sequents are multisets of formulas and boxed sequents:

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

Trees

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \ldots$ are multisets of formulas:



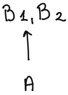
The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

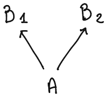
- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a) 

b) 

2) $\Gamma = A, [B_1, B_2, [C], D]$

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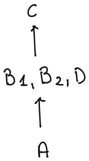
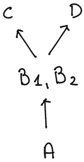
a) 

b) 

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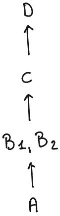
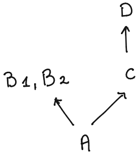
a) 

b) 

Contexts

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

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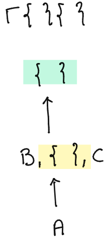
$\Gamma\{\,\}\{\,\} = A, [B, \{\,\}, [\{\,\}], C]$



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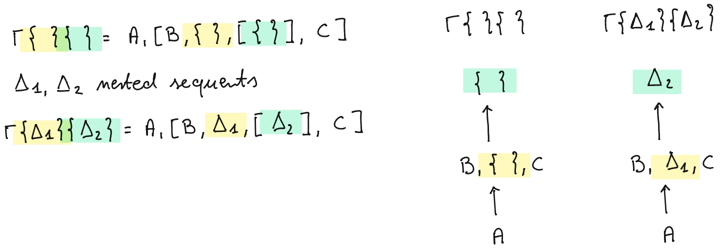

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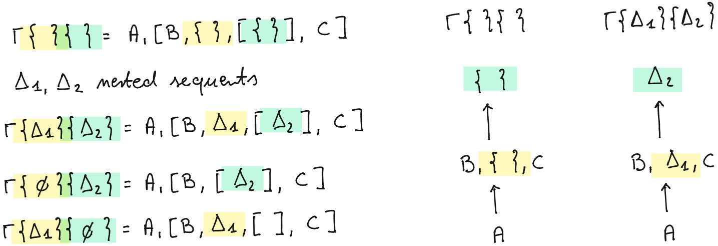

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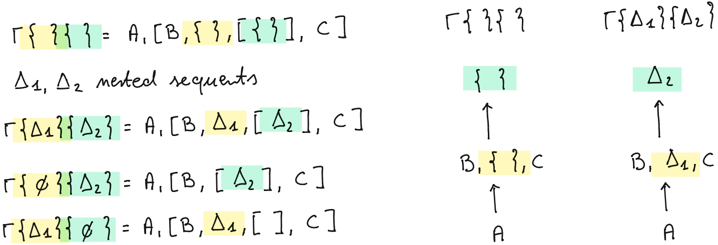

The depth $depth(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $depth(\{\,\}) := 0$;
- $depth(\Gamma\{\,\}, \Delta) := depth(\Gamma\{\,\})$;
- $depth([\Gamma\{\,\}]) := depth(\Gamma\{\,\}) + 1$.

$depth(\Gamma\{\,\}\{\Delta_1\}) = 1$

$depth(\Gamma\{\Delta_1\}\{\,\}) = 2$

Rules of NK

$$\text{init}\ \frac{}{\Gamma\{p, \overline{p}\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

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Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

$$\vee\;\cfrac{\Box\;\cfrac{\vee\;\cfrac{\wedge\;\cfrac{\Diamond\;\cfrac{\text{init}\;\cfrac{}{\Diamond p, [p, \bar{p}, q]}}{\Diamond p, [\bar{p}, q]} \qquad \Diamond\;\cfrac{\text{init}\;\cfrac{}{\Diamond \bar{q}, [\bar{q}, \bar{p}, q]}}{\Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p} \vee q]}}{\Diamond p \wedge \Diamond \bar{q}, \Box(\bar{p} \vee q)}}{(\Diamond p \wedge \Diamond \bar{q}) \vee \Box(\bar{p} \vee q)}$$

Roadmap

HILBERT-STYLE AXIOM SYSTEM

$\Gamma \vdash A$

$\Longleftrightarrow$

LOGICAL CONSEQUENCE

$\Gamma \models A$

$\vdash_{mK} \Gamma \Rightarrow A$

NESTED SEQUENTS (without cut)

Validity of nested sequents [Kuznets & Straßburger, 2018]

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

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$$\mathcal{M}, f(\delta) \models B, \text{ for some } \delta \in tr(\Gamma), \text{ for some } B \in \delta$$

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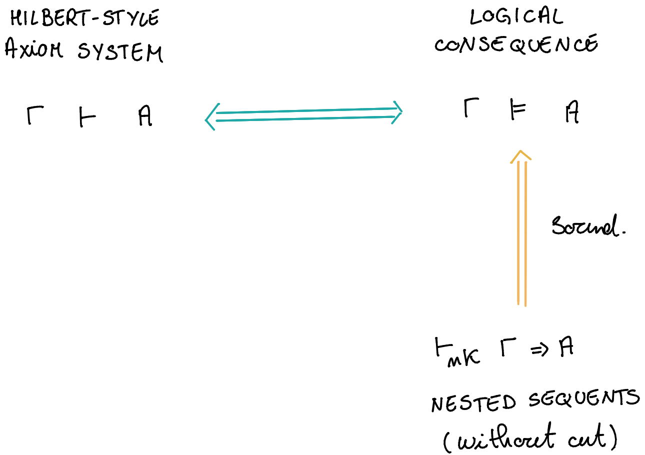
A nested sequent is valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$.

Soundness of NK

Lemma. If Γ is derivable in NK then Γ is valid in all Kripke frames.

Roadmap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(without cut)

Completeness of NK

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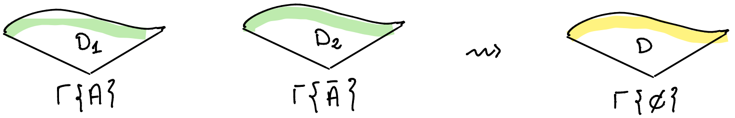
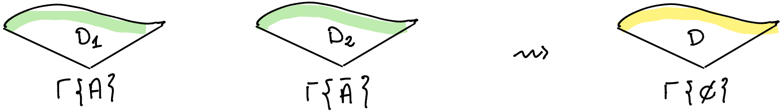
Proof sketch. Assume that the two premisses of cut are derivable in NK, and show how to construct a derivation of the conclusion of the conclusion. Lexicographic induction on (c, h).



One cut reduction case



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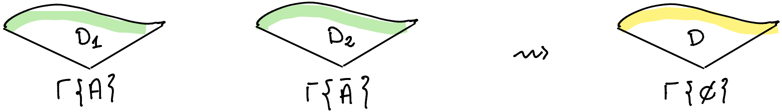

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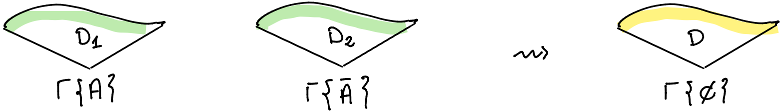

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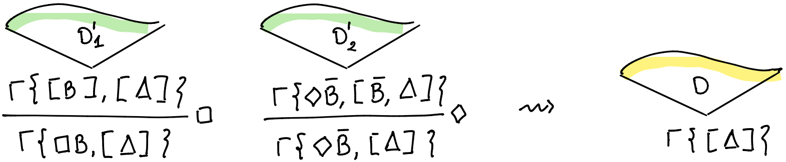

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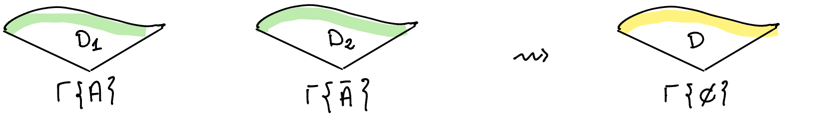

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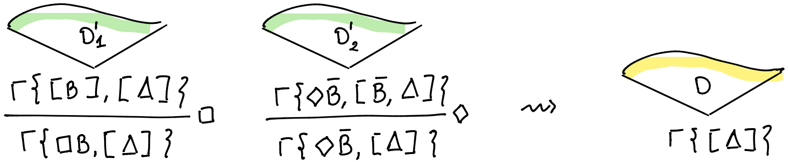
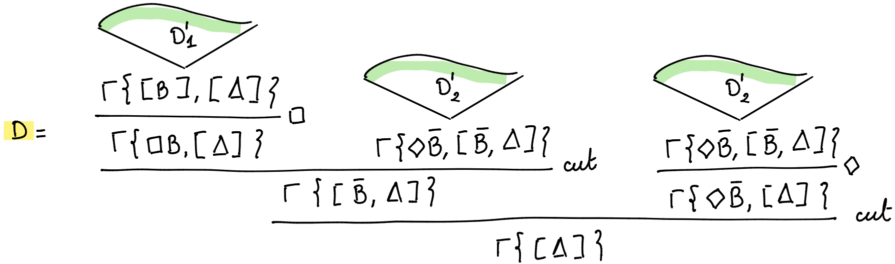
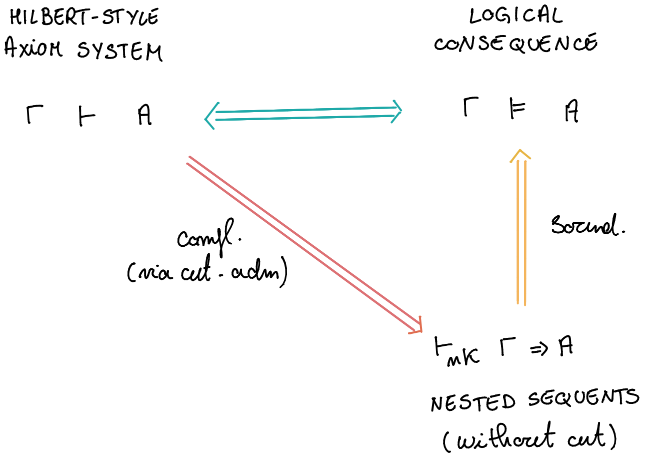




Roadmap

Theorem. If $\Gamma \vdash A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.



Semantic completeness

Lemma (Proof or Countermodel). For Γ nested sequent, either Γ is derivable in NK or there is an $\mathcal{M}$-map for Γ such that Γ is refuted by the $\mathcal{M}$-map.

Theorem (Semantic Completeness). If $\Gamma \models A$, then the nested sequent $\bar{\Gamma} \vee A$ is derivable in NK.

Proof or countermodel

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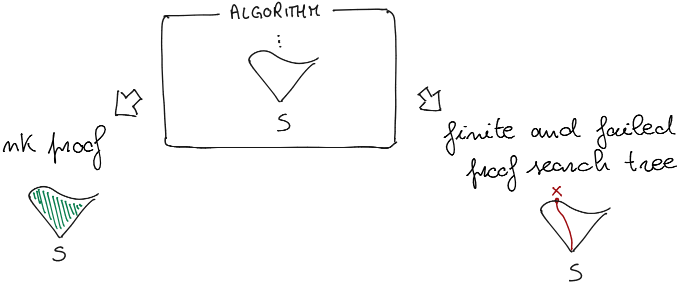
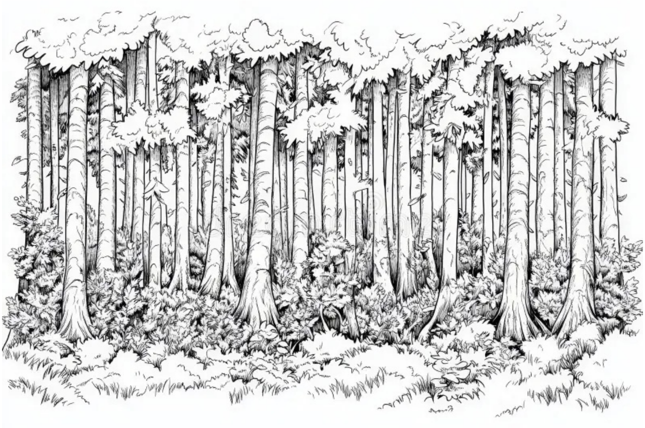
Proof (sketch). Algorithm implementing proof search in nK



Example

$$
\vee\,\dfrac{\vee\,\dfrac{\Box\,\dfrac{\Diamond\,\dfrac{\wedge\,\dfrac{\text{init}\,\dfrac{}{\Diamond(\bar p \wedge \bar q), [\bar p, p], \Box q} \qquad \Box\,\dfrac{\Diamond\,\dfrac{\wedge\,\dfrac{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar p, q] \qquad \text{init}\,\dfrac{}{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar q, q]}}{\Diamond(\bar p \wedge \bar q), [\bar q, p], [\bar p \wedge \bar q, q]}}{\Diamond(\bar p \wedge \bar q), [\bar q, p], [q]}}{\Diamond(\bar p \wedge \bar q), [\bar q, p], \Box q}}{\Diamond(\bar p \wedge \bar q), [\bar p \wedge \bar q, p], \Box q}}{\Diamond(\bar p \wedge \bar q), [p], \Box q}}{\Diamond(\bar p \wedge \bar q), \Box p, \Box q}}{\Diamond(\bar p \wedge \bar q), \Box p \vee \Box q}}{\Diamond(\bar p \wedge \bar q) \vee (\Box p \vee \Box q)}
$$

Nested sequents for the S5-cube



Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^{\diamond}$

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta, \diamond A], A\}}{\Gamma\{[\Delta, \diamond A]\}}$$

$$4^{\diamond}\ \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\Gamma\{\diamond A, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

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Example. Proof of $\Box p \rightarrow \Box\Box p$ in $\mathsf{NK} \cup \{\mathsf{t}, 4\}$

$$\vee\ \cfrac{\Box\ \cfrac{\Box\ \cfrac{4^{\Diamond}\ \cfrac{4^{\Diamond}\ \cfrac{\mathsf{t}^{\Diamond}\ \cfrac{\mathsf{init}\ \cfrac{}{\Diamond\bar{p}, [\Diamond\bar{p}, [\Diamond\bar{p}, \bar{p}, p]]}}{\Diamond\bar{p}, [\Diamond\bar{p}, [\Diamond\bar{p}, p]]}}{\Diamond\bar{p}, [\Diamond\bar{p}, [p]]}}{\Diamond\bar{p}, [[p]]}}{\Diamond\bar{p}, [\Box p]}}{\Diamond\bar{p}, \Box\Box p}}{\Diamond\bar{p} \vee \Box\Box p}$$

Structural rules [Brünnler, 2009]

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

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Proposition. Rule $5^{\Diamond}$ is derivable in $NK \cup \{5_1^{\Diamond}, 5_2^{\Diamond}, 5_3^{\Diamond}\} \cup \{ctr\}$.

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For $X \subseteq \{d, t, b, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-maps for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Theorem. If Γ is derivable in $NK \cup X^{\diamond}$ then Γ is valid in all X-frames.

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- Axiom 5, that is, $\Diamond A \rightarrow \Box\Diamond A$, is valid in all $\{\mathrm{b}, 4\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$.

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Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{b^{\Diamond}, 4^{\Diamond}\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\,b^{\Diamond}}{\Box\bar{p}, [\Diamond p]}\,\Box}{\Box\bar{p}, \Box\Diamond p}\,\Box}{\Box\bar{p} \vee \Box\Diamond p}\,\vee
$$

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- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{t, 5\}$-frames, but it is not derivable in $\mathsf{NK} \cup \{t^{\Diamond}, 5^{\Diamond}\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathsf{NK} \cup \{b^{\Diamond}, 4^{\Diamond}\}$

$$
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$$
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For $X \subseteq \{d, t, b, 4, 5\}$, the 45-closure of X is defined as:

$$\hat{X} = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$

We say that X is 45-closed if $X = \hat{X}$.

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Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

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To prove:

Theorem (Completeness). For $X \subseteq \{d, t, b, 4, 5\}$, if Γ is X-valid, then Γ is derivable in $\mathsf{NK} \cup \hat{X}^{\diamond}$.

Solution # 1 - Syntactic completeness [Brünnler, 2009]

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\diamond}$.

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The proof uses:

- A generalised version of cut (Y-cut, eliminable)

$$\dfrac{\square\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad \mathsf{tr}^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}\,\mathsf{cut}$$

- Additional structural modal rules (admissible)

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Example: 4-cut

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond \overline{A}\}\{\Diamond \overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

If $\mathrm{Y} = \{4\}$, then $\Gamma\{\ \}\{\ \}^n$ is of the form $\Gamma_1\{\{\ \}, \Gamma_2\{\ \}^n\}$:

$$\text{4-cut}\ \frac{\Gamma_1\{\{\Box A\}, \Gamma_2\{\emptyset\}^n\} \quad \Gamma_1\{\{\Diamond A\}, \Gamma_2\{\Diamond A\}^n\}}{\Gamma_1\{\{\emptyset\}, \Gamma_2\{\emptyset\}^n\}}$$

$$\text{cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad 4^{\Diamond}\ \dfrac{\Gamma\{\Diamond \overline{A}, [\Diamond \overline{A}, \Delta]\}}{\Gamma\{\Diamond \overline{A}, [\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \text{4-cut}\ \frac{\Box\ \dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Gamma\{\Diamond \overline{A}, [\Diamond \overline{A}, \Delta]\}}{\Gamma\{[\Delta]\}}$$

cut and Y-cut

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

In the Y-cut:

- $\{\Delta\}^n$ denotes $\overbrace{\{\Delta\}\ldots\{\Delta\}}^{n \text{ times}}$;
- $n \geq 0$;
- $\mathsf{Y} \subseteq \{4, 5\}$;
- there is a derivation of $\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n$ to $\Gamma\{\Diamond\overline{A}\}\{\emptyset\}^n$ in system $\mathsf{Y}^\Diamond$.

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$\Gamma_2\{\ \}^1 = [\{\ \}, \Delta]$

$\Gamma_1\{\ \}\{\ \} = \Gamma\{\{\ \}, [\{\ \}, \Delta]\}$

Structural modal rules

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

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For $X \subseteq \{d, t, b, 4, 5\}$, we write $X^{[\,]}$ for the corresponding subset of $\{d^{[\,]}, t^{[\,]}, b^{[\,]}, 4^{[\,]}, 5^{[\,]}\}$.

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$$\vee\ \dfrac{\Box\ \dfrac{\Box\ \dfrac{\mathsf{b}^{[\,]}\ \dfrac{4^{[\,]}\ \dfrac{\Diamond\ \dfrac{\mathsf{init}\ \overline{}}{[[[\bar{p},p],\Diamond p]]}}{[[[\bar{p}],\Diamond p]]}}{[[[\bar{p}]],\Diamond p]}}{[\bar{p}],[\Diamond p]}}{\Box\bar{p},[\Diamond p]}}{\Box\bar{p},\Box\Diamond p}}{\Box\bar{p} \vee \Box\Diamond p}$$

Cut-admissibility

Theorem (Cut-admissibility). For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ 45-closed, the cut rule and the Y-cut rule are admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

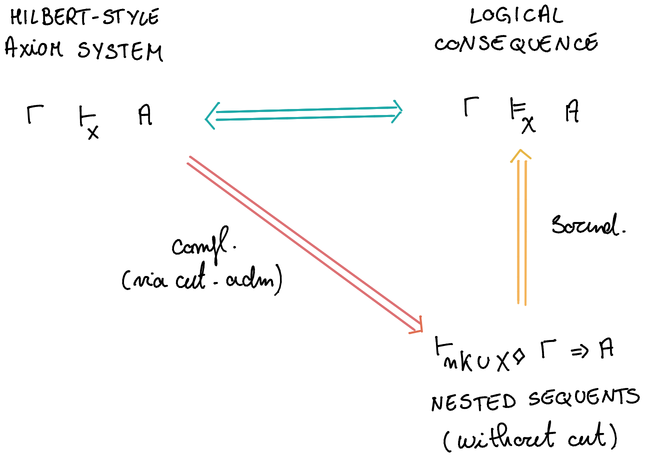
$$
\text{cut}\;\dfrac{\square\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad 4^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}
\quad\rightsquigarrow\quad
4\text{-cut}\;\dfrac{\square\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{[\Delta]\}}
$$

$$
4\text{-cut}\;\dfrac{\square\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\square A,[[\Sigma]]\}} \qquad \Diamond\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}
\quad\rightsquigarrow
$$

$$
\rightsquigarrow\quad
\text{cut}\;\dfrac{4^{[\,]}\,\dfrac{4^{[\,]}\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{[[A],[\Sigma]]\}}}{\Gamma\{[[A,\Sigma]]\}} \qquad 4\text{-cut}\;\dfrac{\text{wk}\,\dfrac{\square\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\square A,[[\Sigma]]\}}}{\Gamma\{\square A,[[\overline{A},\Sigma]]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{[[\overline{A},\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}
$$

Roadmap

$X \subseteq \{d, t, b, 4, 5\}$ and X 45-closed:



Solution # 2 [Marin & Straßburger, 2014]

Can we get rid of the 45-closure condition?

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YES: by adding to NK both the propagation rules $X^{\Diamond}$ and the structural rules $X^{[\,]}$. The price to pay is that contraction is no longer admissible.

Theorem. For $X = \{d, t, b, 4, 5\}$, and Γ a set of formulas, it holds that
Γ is derivable in $NK_{ctr} \cup X^{\Diamond}_{ctr} \cup X^{[\,]}$ iff Γ is X-valid.

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NO, some combinations are incomplete, and one example is given in [Marin & Straßburger, 2014].

Summing up

	fml. interpr.	invertible rules	analyti-city	termination proof search	counterm. constr.	modu-larity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$NK \cup X^{\diamond}$	yes	yes	yes	yes	yes	45-clause

Beyond nested sequents

Other ‘structured’ approaches to define proof systems for modal logics:

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- Hypersequents for S5
  Introduced by: [Mints, 1968], [Pottinger, 1983], [Avron, 1987]
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  A hypersequent $\mathcal{H}$ is a finite multiset of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid .. \mid \Gamma_n \Rightarrow \Delta_n$$

$$\Box_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta} \quad \mathrm{t} \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \quad \Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

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- Display calculi, for (temporal) logics with backward modality
  [Belnap, 1982], [Kracht, 1996], [Wansing, 1994]

Beyond nested sequents

Other 'structured' approaches to define proof systems for modal logics:

- Hypersequents for S5
  Introduced by: [Mints, 1968], [Pottinger, 1983], [Avron, 1987]
  To get started: [Poggiolesi, 2008], [Lellmann, 2016]
  A hypersequent $\mathcal{H}$ is a finite multiset of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid .. \mid \Gamma_n \Rightarrow \Delta_n$$

$$\Box_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta} \quad t \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \quad \Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

- Display calculi, for (temporal) logics with backward modality
  [Belnap, 1982], [Kracht, 1996], [Wansing, 1994]

- Linear nested sequents, lists of sequents [Lellmann, 2015]

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- .. and many more! For an overview: [Lyon et al., 2025]

End of content for today's lecture!

Questions?

Exercises

d	$\Box A \rightarrow \Diamond A$
t	$\Box A \rightarrow A$
b	$A \rightarrow \Box\Diamond A$
4	$\Box A \rightarrow \Box\Box A$
5	$\Diamond A \rightarrow \Box\Diamond A$

1. For $X \in \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, show that the axiom X is derivable in the nested sequent calculus $\mathrm{NK} \cup \mathrm{X}^{\Diamond}$.
2. Show that axiom 4 is valid in all $\{\mathrm{t}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{t}^{\Diamond}, 5^{\Diamond}\}$. Show that the axiom is derivable in $\mathrm{NK} \cup \{\mathrm{t}^{[]}, 5^{[]}\}$.
3. Show that 4 is valid in all $\{\mathrm{b}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 5^{\Diamond}\}$. Show that the axiom is derivable in $\mathrm{NK} \cup \{\mathrm{b}^{[]}, 5^{[]}\}$.
4. Derive axioms t, b and 5 in the hypersequent calculus for S5.