Take-home exam

Proof Theory of Modal Logic
Tsinghua Logic Summer School, July 2025

This exam contains 6 questions, for a total of 20 points. Question 7 is a bonus question, a bit more difficult, which will allow you to gain 3 extra points. The deadline is Monday 21 July, at 23:59. Good luck!

Question 1 (3 points). In this exercises we work with **G3cp**, the sequent calculus for classical propositional logic. Consider the following rule of *converse weakening*:

$$\mathsf{cwk}\,\frac{p, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Is the rule derivable? Is the rule admissible? Motivate your answer to both questions.

Question 2 (3 points). In this exercise we work with the nested sequent calculus NK for modal logic K. Prove that $\Box^{\mathsf{c}}$, the *cumulative* version of rule $\Box$, is admissible in NK. You can use (without proof) admissibility of weakening and contraction in NK, as well as invertibility of all the rules of NK.

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad\qquad \Box^{\mathsf{c}}\,\frac{\Gamma\{\Box A, [A]\}}{\Gamma\{\Box A\}}$$

Is rule $\Box^{\mathsf{c}}$ *height-preserving* admissible in NK? Why?

Question 3 (3 points). Write down the labelled rule den corresponding to the frame condition of *density*, that is:

$$\forall x \forall y \big(xRy \to \exists k (xRk \wedge kRy)\big)$$

Then, derive the formula $\Diamond p \to \Diamond\Diamond p$ in $\mathsf{labK} \cup \{\mathsf{den}\}$.

Question 4 (3 points). We want to show that formula $p \vee \Box(\Box p \to \bot)$ is valid in modal logic $\mathsf{S5}$. Construct a derivation for the formula, using either the labelled calculus $\mathsf{labK} \cup \{\mathsf{ref}, \mathsf{sym}, \mathsf{tr}\}$ or the nested calculus $\mathsf{NK} \cup \{\mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, 4^{\Diamond}, 5^{\Diamond}\}$.

Next, we want to check whether the formula is valid in K. Using either labK or NK, construct a proof of the formula or show that the formula is not derivable in the calculus. In case the formula is not derivable, produce a countermodel for it, that is, find a model $\mathcal{M}$ and a world x such that $\mathcal{M}, x \not\models p \vee \Box(\Box p \to \bot)$ (you can look at the countermodel construction we saw in Lecture 4).

Question 5 (4 points). In this exercise we work with *hypersequents*, a proof system for modal logic $\mathsf{S5}$. We use the language of classical propositional logic with implication but without $\Diamond$, that is:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$$

We set $\neg A := A \to \bot$ and $\Diamond A := \neg\Box\neg A$.

Recall that in the models for S5 the accessibility relation R is a reflexive, transitive and symmetric relation. We shall refer to these models as "S5-models". Hypersequents enrich the *structure* of Gentzen-style sequents by introducing an additional structural connective, $|$, which is interpreted as a disjunction. Formally, a hypersequent $\mathcal{H}$ is a multiset of sequents, that is, the following object where, for $n \geq 0$, and for $i \leq n$, every Γ_i, Δ_i is a multiset of formulas:

$$\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

The rules of the hypersequent calculus are the following[1]:

$$\mathsf{init}\,\frac{}{\mathcal{H} \mid p, \Gamma \Rightarrow \Delta, p} \qquad \bot_{\mathsf{L}}\,\frac{}{\mathcal{H} \mid \bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\,\frac{\mathcal{H} \mid A, B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta}$$

$$\wedge_{\mathsf{R}}\,\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \wedge B} \qquad \vee_{\mathsf{L}}\,\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \quad \mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta}$$

$$\vee_{\mathsf{R}}\,\frac{\mathcal{H} \mid \ \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

$$\to_{\mathsf{L}}\,\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \to B, \Gamma \Rightarrow \Delta} \qquad \to_{\mathsf{R}}\,\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\Box_{\mathsf{L}}\,\frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta} \qquad \mathsf{t}\,\frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathsf{R}}\,\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

We say that a hypersequent $\mathcal{H}$ is S5-*valid* iff there is a sequent $\Gamma \Rightarrow \Delta \in \mathcal{H}$ which is S5-valid. The notion of S5-validity of a sequent is similar to the same we saw in Homework 1, namely: Given a sequent $\Gamma \Rightarrow \Delta$, a S5-model $\mathcal{M}$ and a world x of $\mathcal{M}$, we say that $\Gamma \Rightarrow \Delta$ is S5-*satisfiable at* $\mathcal{M}, x$ (notation: $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$) iff the following holds: *if* for all formulas $G \in \Gamma$ it holds that $\mathcal{M}, x \models G$, *then* there is a formula $D \in \Delta$ such that $\mathcal{M}, x \models D$. We say that $\Gamma \Rightarrow \Delta$ is *satisfiable* iff there are $\mathcal{M}, x$ such that $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$. A sequent $\Gamma \Rightarrow \Delta$ is S5-*valid* iff, for all S5-models $\mathcal{M}$ and for all worlds x, it holds that $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$.

a) Construct proofs for formulas $\Diamond p \to \Box\Diamond p$ and $\Box p \to \Box\Box p$ in the hypersequent calculus for S5.
b) Prove that the hypersequent rule $\Box_{\mathsf{R}}$ is sound, that is: if its premiss is S5-valid, then its conclusion is S5-valid.
c) Suppose that we now add $\Diamond$ as a primitive operator in our language. What rules we would need to add to the hypersequent system to treat $\Diamond$? Write down the rules.
Hint: you can find them by thinking of the definition of $\Diamond$ in terms of $\Box$.

Question 6 (4 points). In this exercise, we wish to establish a translation between the G3-sequent calculus **G3K** and the labelled sequent calculus labK. Both of these are proof systems for modal logic K.

[1] This version of the rules is correct; the version given in the slides of Lecture 2 contains a typo.

First, we define a translation function T_x which, given a label x, maps sequents into labelled sequents. For Γ multiset of formulas, we write $x : \Gamma$ to denote the multiset $\{x{:}G \mid G \in \Gamma\}$. The translation T_x is defined as follows:

$$\mathsf{T}_x(\Gamma \Rightarrow \Delta) = x : \Gamma \Rightarrow x : \Delta$$

In words, the translation 'labels' all the formulas in Γ and Δ with the same label x. So, for instance, $\mathsf{T}_x(A, B \Rightarrow C) = x{:}A, x{:}B \Rightarrow x{:}C$.

Next, we shall prove the following result:

Theorem 1. *If the sequent $\Gamma \Rightarrow \Delta$ is derivable in* **G3K***, then the labelled sequent $\mathsf{T}_x(\Gamma \Rightarrow \Delta)$ is derivable in* labK*.*

The proof proceeds by induction on the height h of the derivation of $\Gamma \Rightarrow \Delta$ in **G3K**. Prove the base case ($h = 0$) and, for the inductive step ($h = n + 1$), prove the case in which the last rule applied in the derivation of $\Gamma \Rightarrow \Delta$ is k:

$$\mathsf{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Recall that $\Box\Sigma = \{\Box S \mid S \in \Sigma\}$. You can use height-preserving admissibility of substitution, weakening and contraction in labK, as well as height-preserving invertibility of all the rules.

Question 7 ($\star$) (3 points). Continuing from **Question 6**, it is possible to establish a translation between the nested calculus NK and the labelled calculus labK. Write down a translation mapping nested sequents into labelled sequents.

Writing down the translation is quite difficult, also because nested sequents are one-sided, while labelled sequents are two-sided. It might help to use the following: for $\Gamma \Rightarrow \Delta$ and $\mathcal{R}', \Gamma' \Rightarrow \Delta'$ labelled sequents, we write $(\Gamma \Rightarrow \Delta) \otimes (\mathcal{R}', \Gamma' \Rightarrow \Delta')$ to denote the labelled sequent $\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

To test whether your translation works, show how to translate an instance of application of rule $\Box$ from NK into the labelled calculus labK (you can ignore contexts).