

Homework 3

Proof Theory of Modal Logic
Tsinghua Logic Summer School, July 2025

Exercises marked with (★) are not mandatory, but allow you to gain extra points.

Question 1 (3 points). Write down the labelled rule **conf** corresponding to the frame condition of *confluence*, that is:

$$\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow \exists k (yRk \wedge zRk))$$

Then, derive the formula $\Diamond \Box p \rightarrow \Box \Diamond p$ in $\text{labK} \cup \{\text{conf}\}$.

Question 2 (3.5 points). Derive axiom 4, that is, $\Box A \rightarrow \Box \Box A$, in $\text{labK} \cup \{\text{t}, 5\}$. Then, show that rule **tr** is derivable in $\text{labK} \cup \{\text{t}, 5\} \cup \{\text{wk}_L, \text{wk}_R\}$.

Recall that $\text{labK} \cup \{\text{t}, 5\}$ consists of the rules of **labK**, plus the following rules (**euc'** is the contracted instance of **euc**):

$$\begin{array}{c} \text{ref} \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{euc} \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{euc}' \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \end{array}$$

The rule **tr** is the following rule:

$$\text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Question 3 (3.5 points). We want to show that the rule \Diamond_L is *invertible*, that is, we want to prove the following statement:

*If $\mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta$ is derivable in **labK**, then for every label $y \neq x$ which does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$, it holds that $xRy, \mathcal{R}, y:A, \Gamma \Rightarrow \Delta$ is derivable in **labK**.*

The proof proceeds by induction on the height of the derivation of $\mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta$, and by distinguishing cases according to the last rule applied in the derivation. Prove the statement for the case in which the last rule applied is \Box_R , that is, the derivation has the following form:

$$\Box_R \frac{\begin{array}{c} \triangle \\ \mathcal{D} \end{array} \frac{zRy, \mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta', y:B}{\mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta', z:\Box B}}{\mathcal{R}, x:\Diamond A, \Gamma \Rightarrow \Delta', z:\Box B}$$

Show how to construct a derivation of sequent $xRy, \mathcal{R}, y:A, \Gamma \Rightarrow \Delta', z:\Box B$. You can use (without proof) admissibility of substitution and weakening, seen in the lecture.

Question 4 (★) (2 points). Show that rule **euc** is derivable in $\mathbf{labK} \cup \{\mathbf{b}, 4\} \cup \{\mathbf{wk}_L, \mathbf{wk}_R\}$.

Recall that $\mathbf{labK} \cup \{\mathbf{b}, 4\}$ consists of the rules of \mathbf{labK} , plus the following rules:

$$\text{sym} \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

The rule **euc** is the following rule:

$$\text{euc} \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Is rule **euc** derivable in $\mathbf{labK} \cup \{\mathbf{b}, 4\}$? Is it admissible in $\mathbf{labK} \cup \{\mathbf{b}, 4\}$? Why?