Homework 1

Proof Theory of Modal Logic
Tsinghua Logic Summer School, July 2025

Exercises marked with ($\star$) are not mandatory, but allow you to gain extra points.

Question 1 (3 points). Consider the following rule efq (*ex falso quodlibet*):

$$\mathsf{efq}\,\frac{\Rightarrow \bot}{\Rightarrow A}$$

Is the rule derivable in **G3cp**? Is the rule admissible in **G3cp**? Motivate your answer to both questions.

Question 2 (3 points). In this exercise, we take $\neg A$ to be a shorthand for $A \to \bot$. Show that the following rules for negation are derivable in $\mathbf{G3cp} \cup \{\mathsf{wk_L}, \mathsf{wk_R}, \mathsf{ctr_L}, \mathsf{ctr_R}\}$.

$$\neg_{\mathsf{L}}\,\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \neg_{\mathsf{R}}\,\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

Are the rules admissible in **G3cp**? Motivate your answer.

Question 3 (4 points). In this exercise, we want to prove soundness of **G3K** with respect to the semantics for K. We start with some definitions.

Given a sequent $\Gamma \Rightarrow \Delta$, a model $\mathcal{M}$ and a world x of $\mathcal{M}$, we say that $\Gamma \Rightarrow \Delta$ is *satisfiable at* $\mathcal{M}, x$ (notation: $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$) iff the following holds: *if* for all formulas $G \in \Gamma$ it holds that $\mathcal{M}, x \models G$, *then* there is a formula $D \in \Delta$ such that $\mathcal{M}, x \models D$. We say that $\Gamma \Rightarrow \Delta$ is *satisfiable* iff there are $\mathcal{M}, x$ such that $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$. A sequent $\Gamma \Rightarrow \Delta$ is *valid* iff, for all models $\mathcal{M}$ and for all worlds x, it holds that $\mathcal{M}, x \models \Gamma \Rightarrow \Delta$. We wish to prove one case of the following result:

Theorem 1. *If* $\vdash_{\mathbf{G3K}} \Gamma \Rightarrow \Delta$*, then* $\models \Gamma \Rightarrow \Delta$.

The proof proceeds by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$. For the base case ($h = 0$) we need to prove that the initial sequents of **G3K** are valid. For the inductive step ($h > 0$), we need to prove that the rules of **G3K** preserve validity, that is: assuming that the premiss(es) of a rule are valid, we need to show that the conclusion of the rule is valid.

Show that rule k preserves validity, that is: if $\models \Sigma \Rightarrow A$ then $\models \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$.

$$\mathsf{k}\,\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Hint: it is easier to prove the contrapositive statement, that is: if $\not\models \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$ *then* $\not\models \Sigma \Rightarrow A$.

Question 4 ($\star$) (2 points). Show that if $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta, \bot$ then $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$. You will need to reason by induction on the height h of the derivation of $\Gamma \Rightarrow \Delta, \bot$. If $h = 0$, then $\Gamma \Rightarrow \Delta, \bot$ is an initial sequent (and you need to show that also $\Gamma \Rightarrow \Delta$ is an initial sequent). If $h > 0$, then $\Gamma \Rightarrow \Delta, \bot$ has been derived by application of some rule R of **G3cp**. Distinguish cases according to the shape of R, applying the inductive hypothesis to the premiss(es) of R. You can treat similar cases at the same time.