Solutions of Homework 3

Q1

$$\frac{yRk, zRk, xRy, xRz, R, \Gamma \Rightarrow \Delta}{xRy, xRz, R, \Gamma \Rightarrow \Delta} \text{ conf}$$

where label k is fresh

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{}\text{ init}}{yRk, zRk, xRy, xRz, z:\Box p, k:p \Rightarrow y:\Diamond p, k:p}\ \Diamond R}{yRk, zRk, xRy, xRz, z:\Box p, k:p \Rightarrow y:\Diamond p}\ \Box_L}{yRk, zRk, xRy, xRz, z:\Box p \Rightarrow y:\Diamond p}\ \text{conf}}{xRy, xRz, z:\Box p \Rightarrow y:\Diamond p}\ \Diamond_L}{xRy, x:\Diamond\Box p \Rightarrow y:\Diamond p}\ \Box R}{x:\Diamond\Box p \Rightarrow x:\Box\Diamond p}\ \to R}{\Rightarrow x:\Diamond\Box p \to \Box\Diamond p}$$

Q2.

Rule tr is derivable:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRz, zRy, yRy, yRx, xRx, xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{wk}_L\ 4x}{zRy\ \ yRy, yRx, xRx, xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{euc}}{yRy, yRx, xRx, xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{euc}}{yRx, xRx, xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{ref}}{xRx, xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{euc}}{xRy, yRz, \Gamma \Rightarrow \Delta}\ \text{ref}$$
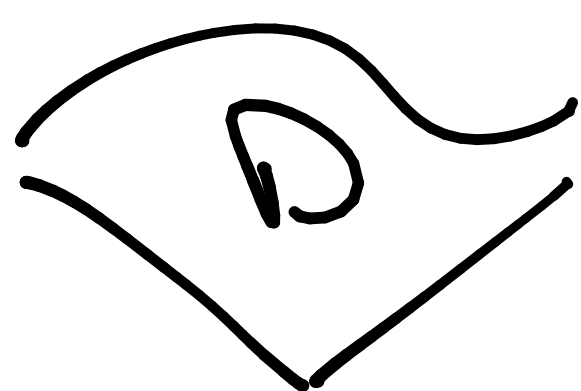
Derivation of $\Box A \to \Box\Box A$

(To be precise, the topmost sequent is not an instance of init but it is derivable! I should have written the axiom as $\Box p \to \Box\Box p$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rule{3cm}{0.4pt}}{xRz, xRz, zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A, z:A \Rightarrow z:A}\ init}{xRz, zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\ \Box_L}{zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\ euc}{yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\ euc}{yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\ ref}{xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\ euc}{xRy, yRz, x:\Box A \Rightarrow z:A}\ ref}{xRy, x:\Box A \Rightarrow y:\Box A}\ \Box_R}{x:\Box A \Rightarrow x:\Box\Box A}\ \Box_R}{\Rightarrow x:\Box A \to x:\Box\Box A}\ {\to}R$$

Q3 Derivation of $xRy, R, y:A \Rightarrow \Delta', z:\square B$

$$\overbrace{D}$$

$$\frac{zRy, R, x:\Diamond A, \Gamma \Rightarrow \Delta', y:B}{zRk, R, x:\Diamond A, \Gamma \Rightarrow \Delta', k:B} \text{ subst } [y/k]$$

$$\frac{zRk, R, x:\Diamond A, \Gamma \Rightarrow \Delta', k:B}{xRy, zRk, R, y:A, \Gamma \Rightarrow \Delta', k:B} \text{ IH}$$

$$\frac{xRy, zRk, R, y:A, \Gamma \Rightarrow \Delta', k:B}{xRy, R, y:A, \Gamma \Rightarrow \Delta', z:\square B} \square_R$$

In the substitution, we need to choose a variable k which does not occur in Γ or Δ', and which is different from y and x (otherwise we cannot apply the IH, nor we can apply rule $\square_R$.

Q4

$$\frac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{yRz, xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ wk}$$

$$\frac{yRz, xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ tr}$$

$$\frac{xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ sym}$$

$$\frac{yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta} \text{ sym}$$