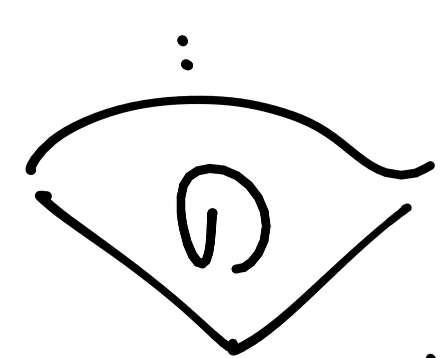
Solutions of Homework 2

Q1. We need to show that, if there is a derivation of $\Gamma\{\Diamond A, [\Delta]\}$ in NK having height n, then there is a derivation of $\Gamma\{\Diamond A, [A, \Delta]\}$ whose height is at most n.

We can use height-preserving admissibility of weakening in NK to construct our desired derivation:



$$\frac{\Gamma\{\Diamond A, [\Delta]\}}{\Gamma\{\Diamond A, [A, \Delta]\}}\ wk$$

The preservation of the height is guaranteed by the fact that weakening is hp-admissible.

Q2. We prove the contrapositive statement: if $\Gamma\{\Diamond A, [\Delta]\}$ is not valid in transitive models, then $\Gamma\{\Diamond A, [\Diamond A, \Delta]\}$ is not valid in transitive models.

Assume $\Gamma\{\Diamond A, [\Delta]\}$ is not valid. Then there is a transitive model $\mathcal{M}$ and a $\mathcal{M}$-map f s.t.:

$\mathcal{M}, f(\delta) \not\Vdash B$, for all $\delta \in tr(\Gamma\{\Diamond A, [\Delta]\})$,
for all $B \in \delta$.

Let γ be s.t. $\Diamond A \in \gamma$ and $\varepsilon = \Delta$. Both γ and ε are nodes of $tr(\Gamma\{\Diamond A, [\Delta]\})$. We have that $\mathcal{M}, f(\delta) \not\Vdash \Diamond A$, and $\mathcal{M}, f(\varepsilon) \not\Vdash D$, for all $D \in \Delta$.

Moreover, since $\mathcal{M}$ is transitive, it holds that $\mathcal{M}, f(\varepsilon) \not\Vdash \Diamond A$. To see this, consider all the worlds $w \in W$ s.t. $f(\varepsilon) R w$. By transitivity of R, we have that $f(\delta) R w$, and so $w \not\models A$. Since this holds for any world accessible from $f(\varepsilon)$, conclude that $\mathcal{M}, f(\varepsilon) \not\models \Diamond A$.

We thus conclude that $\mathcal{M}$ and f refute $\Gamma\{\Diamond A, [\Diamond A, \Delta]\}$

Q3. We distinguish cases according to the form of A.

- $A := p$. Then $\Gamma\{p, \bar{p}\}$ is an instance of init, and we are done.

- $A := B \wedge C$. We construct the following derivation:

$$\dfrac{\dfrac{\Gamma\{B, \bar{B}\} \qquad \Gamma\{C, \bar{C}\}}{\Gamma\{B \wedge C, \bar{B}, \bar{C}\}} \wedge}{\Gamma\{B \wedge C, \bar{B} \vee \bar{C}\}} \vee$$

Since $cp(B) < cp(B \wedge C)$ and $cp(C) < cp(B \wedge C)$, both $\Gamma\{B, \bar{B}\}$ and $\Gamma\{C, \bar{C}\}$ are derivable by inductive hypothesis.

- $A := B \vee C$. Similar to the previous case.

- $A := \Box B$. We construct the derivation:

$$\dfrac{\dfrac{\Gamma\{[B, \bar{B}], \Diamond \bar{B}\}}{\Gamma\{[B], \Diamond \bar{B}\}} \Diamond}{\Gamma\{\Box B, \Diamond \bar{B}\}} \Box$$

The sequent $\Gamma\{[B, \bar{B}], \Diamond \bar{B}\}$ can be written as follows (changing the context):

$\Gamma'\{B, \bar{B}\}$, where $\Gamma'\{\ \} = \Gamma\{[\{\ \}], \Diamond \bar{B}\}$.

Since the statement says "for any context", we can still apply the inductive hypothesis, and conclude that $\Gamma'\{B, \bar{B}\}$ is derivable.

Q4.

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{\Diamond A, [\Delta], [\Diamond A, \Sigma]\}\end{matrix}}{\Gamma\{\Diamond A, [\Delta], [\Sigma]\}}\ 4^{\Diamond}$$

$$\mathcal{D}$$

By IH applied to $\Gamma\{\Diamond A, [\Delta], [\Diamond A, \Sigma]\}$, we have that $\Gamma\{\Diamond A, [\Delta, [\Diamond A, \Sigma]]\}$ is derivable.

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma\{\Diamond A, [\Delta, [\Diamond A, \Sigma]]\}\end{matrix}}{\Gamma\{\Diamond A, [\Delta, \Diamond A, [\Diamond A, \Sigma]]\}}\ wk}{\Gamma\{\Diamond A, [\Delta, \Diamond A, [\Sigma]]\}}\ 4^{\Diamond}}{\Gamma\{\Diamond A, [\Delta, [\Sigma]]\}}\ 4^{\Diamond}$$