Solutions of Homework 1

Q 1. Rule efq is not derivable in G3cp.
This is because there are no rules which allow us to derive sequent $\Rightarrow A$ from $\Rightarrow \bot$.
Rule efq is admissible in G3cp. This is because there is no derivation of $\Rightarrow \bot$ in G3cp, and thus the following condition is vacuously satisfied:

if $\Rightarrow \bot$ is derivable, then $\Rightarrow A$ is derivable

Q 2.

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot_L}{A \to \bot, \Gamma \Rightarrow \Delta}\,\to_L$$

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta, \bot}\,wk_R}{\Gamma \Rightarrow \Delta, A \to \bot}\,\to_R$$

Q 3. We show that:

if $\not\models \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$ then $\not\models \Sigma \Rightarrow A$

Assume $\not\models \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$. Then, there is a model $\mathcal{M}$ and a world x s.t.

$\mathcal{M} = \langle W, R, v \rangle$ $\qquad \mathcal{M}, x \not\models \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta$

that is: [continues on next page]

1) $\mathcal{M}, x \models G$, for all $G \in \Gamma$; and

2) $\mathcal{M}, x \models \Box S$, for all $\Box S \in \Box \Sigma$; and

3) $\mathcal{M}, x \not\models \Box A$; and

4) $\mathcal{M}, x \not\models D$, for all $D \in \Delta$.

From 3), we have that there is a world $y \in W$ s.t. xRy and $y \not\models A$.

Moreover, since $\mathcal{M}, x \models \Box S$, we have that $\mathcal{M}, y \models S$. This holds for all $\Box S \in \Box \Sigma$.

Therefore, we have that

5) $\mathcal{M}, y \models S$, for all $S \in \Sigma$

6) $\mathcal{M}, y \not\models A$

We can thus conclude that $\mathcal{M}, y \not\models \Sigma \Rightarrow A$, whence $\not\models \Sigma \Rightarrow A$.

Q4. By induction on the height of the derivation of $\Gamma \Rightarrow \Delta, \bot$.

<u>$h = 0$</u>. Then $\Gamma \Rightarrow \Delta, \bot$ is an initial sequent. There are two cases:

- $\Gamma \Rightarrow \Delta, \bot$ is of the form

$$\underbrace{\Gamma', p}_{\Gamma} \Rightarrow \underbrace{\Delta', p}_{\Delta}, \bot \quad (p \in \Gamma \cap \Delta)$$

then, also $\Gamma', p \Rightarrow \Delta', p$ is an initial sequent.

▷ $\Gamma \Rightarrow \Delta, \bot$ is of the form

$$\underbrace{\bot, \Gamma'}_{\Gamma} \Rightarrow \Delta, \bot$$

then, also $\bot, \Gamma' \Rightarrow \Delta$ is an initial sequent.

<u>$h = n+1$</u>. All the cases immediately follow by applying IH to the premiss(es) of the last rule R applied in the derivation of $\Gamma \Rightarrow \Delta, \bot$

This is because $\bot$ is never principal in any rule application.

The case distinction is as follows:

▷ R is a one-premiss rule. Then our derivation is:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma' \Rightarrow \Delta', \bot\end{array}}{\Gamma \Rightarrow \Delta, \bot}\, R \quad \xrightarrow{\text{IH}} \quad \boxed{\dfrac{\begin{array}{c}\mathcal{D}^*\\ \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma \Rightarrow \Delta}\, R}$$

This is the deriv. of $\Gamma \Rightarrow \Delta$ we wanted!

▷ R is a two-premisses rule:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow \Delta_1, \bot\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \Rightarrow \Delta_2, \bot\end{array}}{\Gamma \Rightarrow \Delta, \bot}\, R \quad \xrightarrow{\text{IH}} \quad \dfrac{\begin{array}{c}\mathcal{D}_1^*\\ \Gamma_1 \Rightarrow \Delta_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2^*\\ \Gamma_2 \Rightarrow \Delta_2\end{array}}{\Gamma \Rightarrow \Delta}\, R$$

□

observe that the following rule is admissible but not derivable in G3cp: $\dfrac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta}$