

## Solutions of Homework 1

Q1. Rule efq is not derivable in G3cp.

This is because there are no rules which allow us to derive sequent  $\Rightarrow A$  from  $\Rightarrow \perp$ .

Rule efg is admissible in G3cp. This is because there is no derivation of  $\Rightarrow \perp$  in G3cp, and thus the following condition is vacuously satisfied:

if  $\Rightarrow \perp$  is derivable, then  $\Rightarrow A$  is derivable

Q2.

$$\frac{\Gamma \Rightarrow \Delta, A \quad \perp, \Gamma \Rightarrow \Delta}{A \rightarrow \perp, \Gamma \Rightarrow \Delta} \rightarrow_L$$

$$\frac{\frac{A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta, \perp} wk_R}{\Gamma \Rightarrow \Delta, A \rightarrow \perp} \rightarrow_R$$

Q3. we show that :

if  $\not\models \Gamma, \Box \Sigma \Rightarrow \Box A, \Delta$  then  $\not\models \Sigma \Rightarrow A$

Assume  $\not\models \Gamma, \Box \Sigma \Rightarrow \Box A, \Delta$ . Then, there is a model  $\mathcal{M}$  and a world  $x$  s.t.

$$\mathcal{M} = \langle W, R, v \rangle \quad \mathcal{M}, x \not\models \Gamma, \Box \Sigma \Rightarrow \Box A, \Delta$$

that is : [continues on next page]

1)  $\mathcal{M}, x \models G$ , for all  $G \in \Gamma$ ; and

2)  $\mathcal{M}, x \models \Box S$ , for all  $\Box S \in \Box \Sigma$ ; and

3)  $\mathcal{M}, x \not\models \Box A$ ; and

4)  $\mathcal{M}, x \not\models D$ , for all  $D \in \Delta$ .

From 3), we have that there is a world  $y \in W$  s.t.  $x R y$  and  $y \not\models A$ .

Moreover, since  $\mathcal{M}, x \models \Box S$ , we have that  $\mathcal{M}, y \models S$ . This holds for all  $\Box S \in \Box \Sigma$ .

Therefore, we have that

5)  $\mathcal{M}, y \models S$ , for all  $S \in \Sigma$

6)  $\mathcal{M}, y \not\models A$

We can thus conclude that  $\mathcal{M}, y \not\models \Sigma \Rightarrow A$ , whence  $\not\models \Sigma \Rightarrow A$ .

**Q.E.D.** By induction on the height of the derivation of  $\Gamma \Rightarrow \Delta, \perp$ .

$h = 0$ . Then  $\Gamma \Rightarrow \Delta, \perp$  is an initial sequent.

There are two cases:

▷  $\Gamma \Rightarrow \Delta, \perp$  is of the form

$$\underbrace{\Gamma', p \Rightarrow \Delta', p, \perp}_{\Gamma} \quad (\underbrace{p \in \Gamma \cap \Delta}_{\Delta})$$

then, also  $\Gamma', p \Rightarrow \Delta', p$  is an initial sequent.

▷  $\Gamma \Rightarrow \Delta, \perp$  is of the form

$$\underbrace{\perp, \Gamma'}_{\Gamma} \Rightarrow \Delta, \perp$$

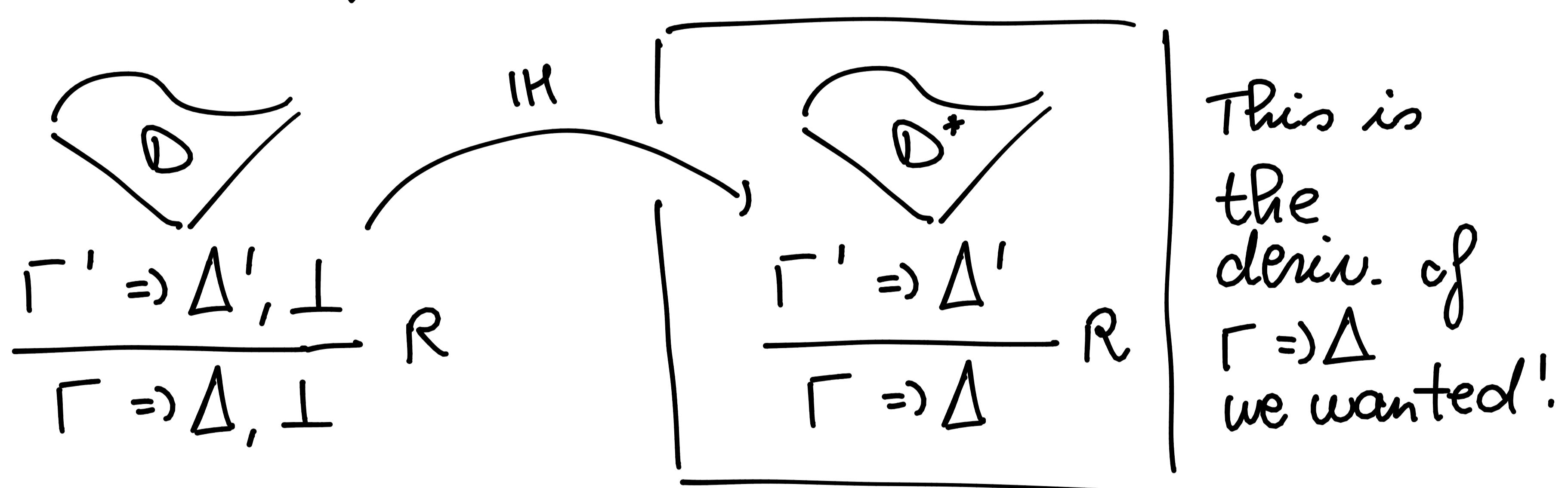
then, also  $\perp, \Gamma' \Rightarrow \Delta$  is an initial sequent.

$h = n+1$ . All the cases immediately follow by applying IH to the premiss(es) of the last rule R applied in the derivation of  $\Gamma \Rightarrow \Delta, \perp$

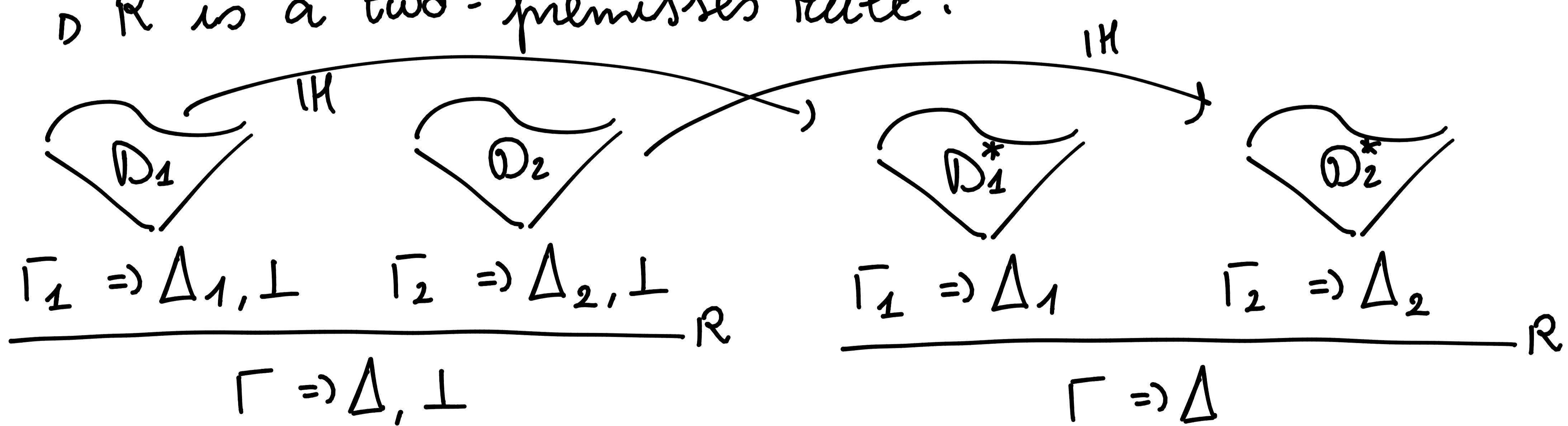
This is because  $\perp$  is never principal in any rule application.

The case distinction is as follows:

▷ R is a one-premiss rule. Then our derivation is:



▷ R is a two-premisses rule:



Observe that the following rule is admissible but not derivable in G3 cp:

$$\frac{\Gamma \Rightarrow \Delta, \perp}{\Gamma \Rightarrow \Delta}$$

□